1. For each of the parts below, state whether the statement is always true, possible, or impossible/false. If your answer is “true” or “false”, explain. If you answer is “possible”, then give more specific conditions indicating when it would always be true. (No credit for one-word answers.)

(a) Two uncorrelated, weakly stationary processes \( \{X(n)\} \) and \( \{Y(n)\} \) have cross spectral density \( S_{XY}(f) = 0 \).
(b) Define \( X(n) = N((n + 1)T) - N(nT) \) where \( T > 0 \) is some fixed and known value and \( N(t) \) is a Poisson process with parameter \( \lambda \). Then \( \{X(n)\} \) is an i.i.d. process with 
\[ C_X(k) = \lambda \delta(k). \]
(c) Let \( \{X(n)\} \) be a discrete-time Markov process. The random variables at two different times \( n \) and \( m \) are dependent for \( n = m + 1 \) but independent if \( n > m + 1 \).
(d) A weakly stationary process with autocorrelation function \( R_X(k) = (-0.5)^{|k|} \) and \( \mu_X = 0 \) has more energy at high frequencies than at low frequencies.

2. A weakly stationary Gaussian process \( \{X(t)\} \) has auto-correlation function \( R_X(\tau) = 5e^{-2|\tau|} \).

For each of the statements below, state whether it is true or false and explain your answer.

(a) There exists an LTI system such that when the input is \( \{X(t)\} \) then the output has a non-zero mean.
(b) If \( \{X(t)\} \) is the input to an LTI system with impulse response \( h(t) = e^{-2t}u(t) \) then the output process is white.
(c) \( Y(t) = X(t)^2 \) is a Gaussian process.
(d) If \( \{X(t)\} \) is the input to an LTI system with impulse response \( h(t) = e^{-t^2} \), then the output process and the input process are orthogonal.
(e) If \( \{X(t)\} \) is the input to a high-pass filter, then the output is a band-pass process.

3. The continuous-time random process \( \{Y(t)\} \) has a zero mean and a power spectral density in the Laplace domain of 
\[ S_Y(s) = \frac{100 - 4s^2}{(9 - s^2)(4 - s^2)}. \]

For each of the statements below, determine whether the answer is true or false and explain why.

(a) \( \{Y(t)\} \) could be generated by the second order system
\[ \frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 10x(t) + 2\frac{dx(t)}{dt}. \]
(b) \( Y(t) \) has variance \( \sigma^2 = 5/6 \).

(c) \( \{Y(t)\} \) can be generated by the system with frequency response

\[
H(f) = \frac{1}{(3 + j2\pi f)}
\]

if the input process has an autocorrelation of the form \( R_X(\tau) = Ke^{-2|\tau|} \).

4. For each process below, you are given some information about a random process. If this information fully specifies the process distribution, then write an expression for \( p_X(t_1) \cdots X(t_k) \) (or \( p_X(n_1) \cdots X(n_k) \) if the process is discrete time) and state whether or not the process is stationary. Otherwise, say that the process distribution is not fully specified, and state whether the process is weakly stationary.

(a) \( \{X(n)\} \) is i.i.d. and \( X(n) \in \{0, 1, \ldots\} \) is geometric with parameter \( q \).

(b) \( \{X(t)\} \) is a Wiener process with \( R_X(t, s) = 0.5 \min(t, s) \).

(c) \( \{X(t)\} \) has power spectral density \( S_X(f) = \frac{2}{1+(2\pi f)^2} \).

(d) \( \{X(n)\} \) is uncorrelated and \( X(n) \) is Laplacian with parameter \( \alpha n \).

(e) \( \{X(t)\} \) is a Markov process where \( p_{X(t) \mid X(s)}(x_t \mid x_s) \) is given by a Poisson distribution with parameter \( \lambda(t - s) \) when \( t > s \).

5. The discrete-time random process \( \{X(n)\} \) has a zero mean and is generated by the system

\[
X(n) = aX(n - 1) + bV(n),
\]

where the input process \( \{V(n)\} \) is weakly stationary and Gaussian. For each of the different conditions below, find \( a \) and \( b \) or explain why it is impossible to find those parameters with the given information.

(a) \( R_X(k) = 4(\frac{1}{3})^{|k|} \) and \( \{V(n)\} \) is zero-mean and white with \( \sigma^2 = 3 \).

(b) \( \{V(n)\} \) is i.i.d. with \( \sigma^2 = 1 \) and \( C_X(0) = 4/3, C_X(1) = -2/3, \) and \( C_X(2) = 1/2 \).

(c) \( S_V(f) = 5/4 - \cos(2\pi f) \) and \( S_X(f) = 1 \).

(d) \( C_V(k) = \delta(k) \) and \( \mu_V = 1 \), and

\[
S_X(z) = \frac{.19}{(1 + 0.9z)(1 + 0.9z^{-1})}
\]

6. For each of the random processes below, state whether it is or is not mean ergodic. Explain.

(a) A zero-mean discrete-time process has a power spectral density of \( S_X(f) = \delta(f) + 2 \) for \( f \in [-0.5, 0.5] \) and is periodic with period 1.
(b) A weakly stationary discrete-time process has $m_X(n) = 0$ and 
\[ R_X(k) = (0.8)^{|k|} \ast (0.5\delta(k-1) + \delta(k) + 0.5\delta(k+1)). \]

(c) The Gaussian process $\{X(t)\}$ has power spectral density $S_X(f) = \frac{5}{(9+(2\pi f)^2)}$ and $m_X(t) = 0$.

(d) The discrete-time process has $m_x(t) = \mu$ and 
\[ \lim_{N \to \infty} E[(\frac{1}{N} \sum_{i=1}^{N} X(i) - \mu)^2] = 0 \]

7. Let $\{X(n)\}$ be a two-sided, strictly stationary and ergodic process (which means it is also ergodic in the mean). For each of the following processes: (i) find the mean $\mu_Y(n)$ in terms of $\mu_X$, (ii) determine whether the process is strictly stationary, and (iii) determine whether the process is ergodic in the mean. For (ii) and (iii), be sure to explain your answer.

(a) $Y(n) = \frac{1}{5} \sum_{k=-2}^{2} X(n-k)$

(b) $Y(n) = X(n)$ if $A > 5$ and $Y(n) = 0$ if $A \leq 5$, where $A$ is a Poisson random variable with parameter $\lambda$. $A$ and $\{X(n)\}$ are independent.

8. Let $\{Y(t)\}$ be the output of a linear, time-invariant system when the input process $\{X(t)\}$ is weakly stationary and has $C_X(\tau) = 4\delta(\tau)$. When the input is zero mean, the power spectral density of the output process is (in the Laplace domain) 
\[ S_Y(s) = \frac{16 - 4s^2}{9 - 10s^2 + s^4} \]

(a) Find the system function $H(s)$ for the filter that generated this output process.

(b) Find the autocovariance function of the output process.

9. $W(n)$ is the input to a series of three stable, linear time-invariant, discrete-time systems as follows:
\[ W(n) \to S_1 \to X(n) \to S_2 \to Y(n) \to S_3 \to Z(n) \]

You are given that: i) the input process $\{W(n)\}$ is stationary and Gaussian; ii) the process $\{X(n)\}$ is uncorrelated and has variance 1; and iii) the process $\{Y(n)\}$ has zero mean.

(a) The first system $S_1$ is described by the equation 
\[ X(n) = \sum_{i=0}^{q} a_i W(n-i) \]
Is the input process $\{W(n)\}$ AR, MA or ARMA? what order? Explain.
(b) Given $m_X(n) = 1$ and that the second system $S_2$ is first order of the form $Y(n) = X(n) + \ldots$, find the system frequency response $H_2(f)$.

(c) You know that $C_Z(k)$ is infinite length, $C_Z(k) = -0.4C_Z(k-1)$ for $k > 1$, $C_Z(0) = 1.2$, and $C_Z(1) = 0.36$. Is the process $\{Z(n)\}$ AR, MA, or ARMA? What do you know about system $S_3$ given what you found in part (b)?

10. Let $\{X(t)\}$ be a weakly stationary, zero-mean process, with $S_X(f) = 0$ for $|f| > B$. Define

$$Y_1(t) = X(t) \cos(2\pi f_1 t + \theta_1)$$
$$Y_2(t) = X(t) \cos(2\pi f_2 t + \theta_2)$$

Assume $\theta_1$ and $\theta_2$ are independent of $\{X(t)\}$.

(a) Assume $\theta_1$ and $\theta_2$ are independent and both are uniformly distributed over $[0, 2\pi]$.

i. Under what conditions on $f_1$ and $f_2$ are $\{Y_1(t)\}$ and $\{Y_2(t)\}$ orthogonal? (Hint: go back to the basic definition of orthogonality to solve the problem.)

ii. Sketch $S_Z(f)$ for $Z(t) = Y_1(t) + Y_2(t)$ assuming the processes are orthogonal.

(b) Now assume that $\theta_1 = \theta_2$, i.e. the phases are exactly the same.

i. Find $E[Y_1(t)Y_2(t)]$. (Recall that $\cos(\alpha)\cos(\beta) = 0.5[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$.)

ii. Are the two processes $\{Y_1(t)\}$ and $\{Y_2(t)\}$ jointly weakly stationary?

11. A random variable $M$ is Gaussian with known parameters $\mu$ and $\sigma_0^2$. If the value of $M = m$ is given, then the random process $\{X(i)\}$ is i.i.d. and Gaussian with mean $m$ and known, constant variance $\sigma^2$.

(a) Find the mean and autocorrelation of the process $\{X(i)\}$.

(b) In what sense, if any, is this process stationary?

(c) What does the sample average

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X(i)$$

converge to?

(d) Is $W(n) = \sum_{i=1}^{n} X(i)$ an independent increment process?