Announcements

HW 4 due on Friday due to exam conflicts
Exam review 10/30
Sample exam posted this week
Solutions posted Monday

Goals for today

Quick RP review
Moments for 2 RP's
Simple Functions of RP's
Review of RP's

Give a random experiment and a mapping of the outcome to a signal.

We can characterize low order marginals with equivalent events, just like for random vectors.

Sometimes we are happy just finding 1st and 2nd order moments (using general formula for expectation).

If we want to get higher order distributions, or if we want to characterize the whole process (i.e., characterizing all possible $n^{th}$ order marginals)

→ need a recipe!

Special classes of processes
Specifying $P$ for an entire process is possible when there's a general “recipe” for computing the probability distribution for the random variables associated with an arbitrary collection of times. Some examples:

- **Independent and identically distributed (i.i.d.) processes (discrete-time),** also called a discrete-time memoryless process.

  $P_{X(n_1)X(n_2)...X(n_d)}(x_1, x_2, \ldots, x_d) = \prod_{i=1}^{d} P_X(x_i)$ where $P_{X(k)}(x) = P_X(x) \forall k$

  1st and 2nd order moments:

  $m_X(n) = E[X(n)] = m_X$

  $\mu_X(n, m) = E[X(n)X(m)] = \begin{cases} m_X^2 & n \neq m \\ E[X_n^2] & n = m \end{cases}$

  $\Sigma_{nm} = E[X(n)X(m)] - m_X^2 = \begin{cases} 0 & n \neq m \\ \sigma_X^2 & n = m \end{cases}$

- Gaussian processes can be specified by $m_X(t)$ and $C_X(t_1, t_2)$

  $$\begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_d) \end{bmatrix} \sim N\left(m, \Sigma\right)$$

  $m = \begin{bmatrix} m_X(t_1) \\ m_X(t_2) \\ \vdots \\ m_X(t_d) \end{bmatrix}$

  $\Sigma_{ij} = C_X(t_i, t_j)$

- **Independent increment process (discrete or continuous time)**

  $P_{X(t_1)X(t_2)...X(t_d)}(x_1, x_2, \ldots, x_d) = P_{X(t_1)}(x_1) \prod_{i=2}^{d} P_{W_i}(w_i)$

  Assuming WLOG that $\tau_i = t_i - t_{i-1} > 0$ and $W_i = X(t_i) - X(t_{i-1})$ (and $w_i = x_i - x_{i-1}$)

  **Examples later today**
Moments for RVs vs. RPs

**Random Vector**

- $R_X = E[XX']$ auto-correlation
- $R_X = \Lambda$ (diagonal) orthogonal
- $R_{XY} = E[XY']$ cross-correlation
- $R_{XY} = 0$ uncorrelated matrix

**Random Processes**

- $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$ scalar function
- $X_{orthogonal}$
- $X_{uncorrelated}$
- $C_X = R_X - m_Xm_X'$ auto-covariance
- $C_X = \Lambda$ (diagonal) vector covariance
- $C_{XY} = R_{XY} - m_Xm_Y'$ cross-covariance
- $C_{XY} = 0$ uncorrelated symmetric

May or may not be symmetric

Symmetric square matrix

$e.g. R_X = R_X^T$

Independent $\rightarrow$ Uncorrelated

Uncorrelated & zero-mean $\rightarrow$ orthogonal

Note that if $\begin{bmatrix} X(t) \end{bmatrix}$ & $\begin{bmatrix} Y(t) \end{bmatrix}$ are vector processes, then $R_X(t_1, t_2)$ is a matrix $dx \times dx$.
Example:

\{X(n)\} and \{Y(n)\} are i.i.d. Bernoulli processes with parameters \(p\) and \(q\), respectively. The two processes are mutually independent.

\[ W(n) = X(n) \oplus Y(n) \]

\[ W(n) \in \{0, 1\} \]

\[ \mathbb{E}[W(n)] = P(W(n) = 1) \]

\[ = P(X(n) = 1, Y(n) = 0) + P(X(n) = 0, Y(n) = 1) \]

\[ = p(1-q) + (1-p)q = p + q - 2pq \]

Marginal:

\[ P(W(n) = 1) = p + q - 2pq \]

\[ P(W(n) = 0) = 1 - P(W(n) = 1) = 1 - p - q + 2pq \]

Is \( W(n) \) identically distributed?

Yes, no dependence on \( n \) for first order marginal.

Is it i.i.d.?

Yes

\( W(n) \) is made up of \( X(n) \& Y(n) \)

\( W(j) \) is made up of \( X(j) \& Y(j) \)

\( X(n) \) indep of \( X(j) \& Y(j) \)

\& \( Y(n) \) are
Example: Signal in Noise

\[ Y(n) = aX(n) + N(n) \]

- \( Y(n) \) is a random field
- \( X(n) \) is a deterministic scalar
- \( a \) is a constant
- \( N(n) \) is an i.i.d. zero-mean Gaussian with variance \( \sigma_N^2 \)

Assume noise and signal are independent, and that the noise is i.i.d. with zero mean.

Find moments given \( m_X(n) \), \( R_X(n,m) \)

\[ m_Y(n) = E(Y(n)) = E(aX(n) + N(n)) \]
\[ = a \cdot E(X(n)) + E(N(n)) = a \cdot m_X(n) \]

\[ R_Y(n,m) = E((aX(n) + N(n))(aX(m) + N(m))) \]
\[ = a^2 E(X(n)X(m)) + aE(X(n)N(m)) + aE(X(m)N(n)) + E(N(n)N(m)) \]
\[ = a^2 R_X(n,m) + a \cdot 0 + a \cdot 0 + E(N(n)N(m)) \]
\[ = a^2 R_X(n,m) + \sigma_N^2 \delta_{nm} \]

\[ C_Y(n,m) = R_Y(n,m) - m_Y(n)m_Y(m) \]
\[ = a^2 R_X(n,m) - a^2 m_X(n)m_X(m) + \sigma_N^2 \delta_{nm} \]
\[ = a^2 C_X(n,m) + \sigma_N^2 \delta_{nm} \]
\[ Y(n) = a X(n) + N(n) \]

Find the cross-covariance \( C_{yx}(n, m) \)

\[
C_{yx}(n, m) = E \left[ Y(n) X(m) \right] - m_Y(n) m_X(m)
\]

\[
= E \left[ (aX(n) + N(n)) X(m) \right] - (a m_X(n)) m_X(m)
\]

\[
= a R_x(n, m) + a E(N(n) X(m)) - a m_X(n) m_X(m)
\]

\[
= a \left( R_x(n, m) - m_X(n) m_X(m) \right) = a C_x(n, m)
\]

Note difference from auto-cov.

\[
C_y(n, m) = a^2 C_x(n, m) + \sigma_n^2 \delta_{nm}
\]

\[
C_y(n, m) = C_y(m, n) \text{ Symmetric}
\]

Note \( C_{yx}(n, m) \neq C_{yx}(m, n) \) not necessarily

but \( C_{yx}(n, m) = C_{xy}(m, n) \) not necessarily, but in this case it is
Example: Accumulator process

\[ S_n = \sum_{i=1}^{n} X_i \text{ where } \{X_i\} \text{ is i.i.d.} \]

\[ \mathbb{E}^+ = \{1, 2, 3, \ldots\} \]

\[ M_S(n) = \mathbb{E}(S_n) = \sum_{i=1}^{n} \mathbb{E}(X_i) = n \mu_X \]

\[ R_S(n,m) = \mathbb{E}(S_n S_m) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{E}(X_i X_j) \]

**Assume WLOG (without loss of generality)**

\[ n < m \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{E}(X_i X_j) \]

\[ = \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{i=1}^{n} \sum_{j=i+1}^{m} \mathbb{E}(X_i) \mathbb{E}(X_j) \]

\[ = n \mathbb{E}(X_i^2) + (nm - n) \mu_X^2 \]

\[ C_S(n,m) = R_S(n,m) - m_S(n)m_S(m) \]

\[ = n (\mathbb{E}(X_i^2) - \mu_X^2) + nm \mu_X^2 - nm \mu_X^2 \]

\[ = n \sigma_X^2 \]

In general

\[ C_S(n,m) = \sigma_X^2 \min(n, m) \]

\[ \Rightarrow \text{Var}(X_i) \]
Example: Counting process

\[ S_n = \sum_{i=1}^{n} X_i \text{ where } \{X_i\} \text{ is Bernoulli} \]

Fix \( n \), what is \( P_{S(n)}(k) \)? Binomial with \( q \) \& \( n \)

\[
M_{S(n)} = nq
\]

\[
C_S(n, m) = q(1-q) \min(n, m)
\]

\[
C_S(n, n) = \text{Variance of } S(n)
\]

What if we wanted

\[ P(\text{S}(n_1)S(n_2)\cdots S(n_5) | (s_1, s_2, \ldots, s_5)) \]

\[
= P(S(n_1)), P(S(n_2)), P(S(n_3)), P(S(n_4)), P(S(n_5)) = X_1^*, \ldots, X_5^*
\]

\( W(1) = S(n_1) \) is a Bernoulli with \( q \)

\[ W(2) = S(n_2) - S(n_1) = X_2 + X_3 \quad \text{Binomial with } q \& n = 2 \]

\[ W(3) = S(n_3) - S(n_2) = X_4 \quad \text{Bernoulli w/ } q \]

\[ W(4) = S(n_4) - S(n_3) = X_5 + X_6 + X_7 \quad \text{Binomial w/ } q \& n = 3 \]

\[ W(5) = \sum_{i=8}^{15} X_i \quad \text{Binomial w/ } q \& n = 8 \] (equiv events)

\[ P(\text{S}(1), \text{S}(3), \text{S}(4), \text{S}(7), \text{S}(15)) = P(W(1), W(2), W(3), W(4), W(5)) \]
\[
S_t \text{ is an independent increment process, so you can specify any } n\text{th order marginal in this way}
\]

\[
P(s_1, s_2, \ldots, s_d) = P(s_1 = s_i, s_2 - s_1 = s_2 - s_i, \ldots, s_d - s_{d-1} = s_d - s_{d-1})
\]

\[
= \prod_{i=1}^{d} P(w_i) \quad w_i = s_i - s_{i-1}
\]
Example: Random walk

\[ W_n = \sum_{i=1}^{n} X_i \] where \( \{X_i\} \) is i.i.d., \( x_i \in \{-1, 1\} \)

\[
\begin{align*}
E[W_n] &= n M_x \\
&= n(2q-1) \\
M_x &= 1 \cdot q + (-1)(1-q) = 2q - 1
\end{align*}
\]

\[
Cov(n, m) = \sigma_x^2 \min(n, m) = 4q(1-q) \min(n, m)
\]

This is an indep
increment process,
so we can find the n\textsuperscript{th} order marginals.

If I go k steps forward & n-k steps back, the prob is given by the binomial

\[
P(S_n = 2k - n) = \binom{n}{k} q^k (1-q)^{n-k}
\]
FIGURE 9.7
(a) Random walk process with $p = 1/2$. (b) Four sample functions of symmetric random walk process with $p = 1/2$. (c) Four sample functions of asymmetric random walk with $p = 3/4$. 