EE512 Graphical Models | **University of Washington**
Spring 2006 | **Dept. of Electrical Engineering**

Homework 2: Due Friday, April 14, 2006, 5:00pm

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Problem 1. Consequences of independence

You are given a distribution on $N+1$ variables, $X_{1:N}$ and C. The following conditional independence (CI) assumptions hold: $X_i \perp\!\!\!\perp X_j | C$ for all $i \neq j$, and $X_i \perp\!\!\!\perp X_j$ for all $i \neq j$. Clearly, one such (uninteresting) distribution family satisfying this set of CI constraints is the one where all $N+1$ variables are mutually independent. Your problem is to (1) determine if any *other* distributions exist (either a family or a particular instance) that satisfy these constraints. This means either a) prove that no such distribution exists, or b) if it does (they do) exist, characterize it (them) as fully as possible. (2) If you have found such distribution(s), can it (they) be described either by Bayesian networks (BNs), Markov random fields (MRFs), or factor graphs?

Problem 2. Independence and factorization

In class, we defined conditional independence $X \perp\!\!\!\perp Y | Z$ if it is the case that

$$f(x, y|z) = f(x|z)f(y|z)$$

for all x, y and for all z where $f_Z(z) > 0$.

Using this, (1) prove that $X \perp\!\!\!\perp Y | Z$ if and only if there exists functions g and h such that

$$f(x, y, z) = g(x, z)h(y, z)$$

for all x and y and for all z with $f_Z(z) > 0$. (note that because of this equivalence, conditional independence is often defined in this way).

(2) Must the functions g and h be unique in your proof? Explain why or why not.

Problem 3 Bayes Ball

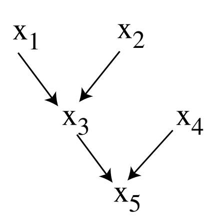

Figure 2: Using Bayes ball, show $X_1 \perp\!\!\!\perp X_2$

Many people do not like d-separation and think that it is easier to use the Bayes Ball procedure/algorithm to determine conditional independence statements in BNs. Others prefer d-separation.

Using the Bayes-ball procedure outlined in class and described more and described in the text (and in Shachter's paper on the web), show that $X_1 \perp\!\!\!\perp X_2$ in the following graph. This means that you must show there is no way for a ball to bounce from X_1 to X_2. This also means you must consider the possibility that the ball bounces below X_3 and then back.

After you've solved this problem, indicate if you prefer Bayes Ball or d-separation (this part is of course not graded, results will be tallied and presented to the class when the HW is graded).

Problem 4. Moralization

In class, we defined the moralization step as a step that we must perform to convert from a BN into an MRF before we perform (variable) elimination. We stated that moralization was crucial because 1) it keeps the purely graphical elimination procedure from doing something that does not corresponds to summation in directed models (i.e., so that we don't break apart or factorize $p(c|\pi_c)$ in invalid ways in general), and 2) it keeps the MRF from expressing conditional independent properties that the Bayesian network (BN) would not state, but at the cost of loosing some of the independence statements that the BN does state.

Regarding conditional independence:

1a) give an example of what a BN would state that the moralization step looses.

1b) give an example of what an MRF would state that would be wrong w.r.t. the BN if moralization was not done.

1c) Prove that after Moralization, no CI statements of a BN are violated by the MRF resulting from the moralized BN. I.e., here you need to prove that the MRF obtained by the moralized BN makes no additional conditional independence statement that is not stated by the original BN.

Problem 5. V-structures and independence

This problem is on V-structures, where we have that $A \rightarrow B \leftarrow C$ meaning that $A \perp\!\!\!\perp C$ but it is not the case that $A \perp\!\!\!\perp C | B$.

3a) Come up with a V-structure parameterization such that $A \perp\!\!\!\perp C | B$. Clearly express your parameterization in terms of numeric values contained in conditional probability tables and for these numeric values, prove that the CI statement holds.

3b) If possible, come up with a V-structure parameterization but where it is such that A is not independent of C, again providing numeric values in the tables. If it is not possible, prove this is the case.

3c) Come up with a V-structure parameterization such that it is not the case that $A \perp\!\!\!\perp C | B$ in general, but for some values of B (say b_i) we have that $A \perp\!\!\!\perp C | B = b_i$. Again, clearly express your parameterization in terms of tables, and prove that the CI statement holds.

Problem 6:

Prove that Bayesian networks that contain no V-structures and over N variables $X_{1:N}$ represent exactly the same family of distributions as MRFs over N variables $X_{1:N}$, where the structures of the BN and MRF are the same, but where the arrow directions are dropped to go from BN to MRF.

Problem 7: Trees

In class, we gave the statement that trees are maximally cycle-free, and that trees are minimally connected. Explain in your own words what this statement means.

Problem 8:

Come up with and draw two MRFs over N variables, where $N >= 10$, where each node is connected to at least 3 other nodes. In one of the cases, your MRF must have a decomposition tree, and draw this tree. In the second case, prove that your graph does not have a decomposition tree.