EE512 Graphical Models | **University of Washington**
Spring 2006 | **Dept. of Electrical Engineering**

Homework 3: Due Friday, April 21, 2006, 5:00pm

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Problem 1. Triangulated graphs and decomposition trees.

1a): Can an underlying triangulated graph always be recovered from a tree decomposition and is this triangulated graph unique? If yes, prove it. If it no, give a counter-example.

1b): Can an underlying triangulated graph always be uniquely recovered from a Junction Tree and if so, will it be unique? If yes, prove it. If it no, give a counter-example.

Problem 2. Triangulation heuristics and perfect elimination orders.

Maximum cardinality search and the minimum fill heuristic will always generate a perfect elimination order if it exists. Give an example that shows that the minimum size heuristic does not have this property.

Problem 3. Minimal separators

3a) For a general graph with $n = |V|$ nodes, come up with a worse case lower bound for the number of minimal separators as a function of n. In other words, demonstrate that there exists at least one graph that has at least $\zeta(n)$ many minimal separators, where $\zeta(n)$ is your bound. Clearly demonstrate how you achieved this bound by giving graphical examples.

3b) Do the same thing as part 3a, but use only the class of triangulated graphs.

Problem 4.

In class, we explained how it made sense to talk about conditional independence relationships like $\{X_1, X_2\} \perp\!\!\!\perp Y | \{Z, X_2\}$. In this problem, either explain how it makes sense, or how it does not make sense to discuss a relationship of the form $\{X_1, X_2\} \perp\!\!\!\perp X_2$.

Problem 5.

In class we showed that (F) $\Rightarrow$ (G) $\Rightarrow$ (L) $\Rightarrow$ (P), but we said it was not the case in general that (L) $\Rightarrow$ (G), nor was it that (P) $\Rightarrow$ (L). Let $\mathcal{P}_0$ be the proposition "(L) $\Rightarrow$ (G)" and $\mathcal{P}_1$ be the proposition "(P) $\Rightarrow$ (L)". Consider probability distributions p_0, p_1, p_2, p_3 over as many variables as you wish where:

- For distribution p_0 it must be the case that both $\mathcal{P}_0$ and $\mathcal{P}_1$ are false.
- For distribution p_1 it must be the case that $\mathcal{P}_0$ is true and $\mathcal{P}_1$ is false.
- For distribution p_2 it must be the case that $\mathcal{P}_0$ is false and $\mathcal{P}_1$ is true.
- For distribution p_3 it must be the case that both $\mathcal{P}_0$ and $\mathcal{P}_1$ are true.

Your task is to find examples of any three out of the above four possibilities, and in each case show why it holds.

Problem 6.

Prove that every triangulated graph of at least $n \geq 2$ nodes has at least two distinct simplicial vertices.

Problem 7.

During the maximum cardinality search (MCS) procedure, one way to try to triangulate the graph (if it is not already triangulated) might be to complete π_i during each step of MCS. Is this guaranteed to triangulate the graph? If so, prove it. If not come up with a counterexample where this procedure will not yield a triangulated graph (note that class notes on this topic have been updated on the web).

Problem 8.

In class, we stated that if $A \perp\!\!\!\perp B | C$ then $A' \perp\!\!\!\perp B' | C$ where $A' \subseteq A$ and $B' \subseteq B$.

8a) First, suppose that $|A| = |B|$, meaning the sizes of the sets are the same. Suppose also that there is some ordering of A, B, lets say $((A_i, B_i)_{i=1}^{|A|})$ (where all of A_i, and B_i are scalars) such that $A_i \perp\!\!\!\perp B_i | C$ for all i. Does it follow that $A \perp\!\!\!\perp B | C$? If so, prove it. If not, find a counterexample.

8b) Next, suppose that it is true that $A_i \perp\!\!\!\perp B_i | C$ for all $A_i \subset A$ and $B_i \subset B$. Does this now imply $A \perp\!\!\!\perp B | C$? Again, either prove or find a counterexample.