EE595A Introduction to Information Theory | **University of Washington**
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Handout 4: Problem Set 2

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DUE: Wednesday, Feb 4th, 2004, in class

Book Problems Do problems 2.22,2.28,3.2,3.3,3.5,12.3 in C&T.

Problem 1:

Let S be a random variable with a binomial distribution with parameters n and p so that

$$P(S = i) = \binom{n}{i} p^i (1-p)^{n-i}$$

show that the most likely value of S is the value $<np>$, where $<np>$ is the integer closest to np. **Hint:** use the fact that

$$\frac{P(S = i+1)}{P(S = i)} = \frac{(n-i)p}{(i+1)(1-p)}$$

and consider the cases when $i < np$, and $i > np$.

Problem 2:

Consider a sphere of radius r in N dimensions. Show that the fraction of the volume of the sphere that lies at values of the radius between $r - \epsilon$ and r, where $0 < \epsilon < r$, is:

$$f = 1 - \left(1 - \frac{\epsilon}{r}\right)^N .$$

Using matlab, evaluate f for the cases $N = 2, N = 10$, and $N = 1000$ with a) $\epsilon/r = 0.01$; and b) $\epsilon/r = 0.5$. Notice that points that are uniformly distributed in a sphere in N dimensions are very likely to be in a thin shell near the surface when N is large. Note that you might find useful the volume of a hypersphere of radius r in N dimensions, which is:

$$V(r, N) = \frac{\pi^{N/2}}{(N/2)!} r^N$$

What is the similarity between this problem and the AEP?

Problem 3: Matlab and AEP

1. Write a matlab function that generates a random variable S according to the binomial distribution with parameters N and p.
2. Write another function that takes the outcome of a binomial random variable and returns the probability of a sequence of Bernoulli trials with parameter p and with that many ones.
3. Write a matlab 'typical' function $A(N, p, \epsilon, S)$ that determines if a sequence with S ones corresponds to a sequence that is a member of the typical set $A_\epsilon^{(N)}$ for a length-N sequence of Bernoulli trials with parameter $P(X = 1) = p$.
4. Starting with N small and with increasing N , show plots depicting the proportion of occurrences (out of some large number, say 10000 or more) that sequences of length N are typical. You should find that when N is small,

for a given ϵ, you get atypicality much more often than when N is large. Do this for the values of ϵ 0.25, 0.1, and 0.001, for the values of p 0.1, 0.3, and 0.5, and for values of N ranging from 1 to 1,000,000 stepping by factors of $\sqrt{10}$ (for larger values of N this program might run for a while). You might also find it helpful to compute log probabilities in places.

Intuitively explain in words the behavior of your 'typical' function when you give it $p = 0.5$ and why this occurs.

In all of the above problems, include all source code. Make sure to clearly segment the code into the different functions, and to accurately document the functions.

Problem 4: Types and type classes

Let the alphabet of a source be $X = a, b, c$ and consider all strings of symbols of length 4 from the source.

- (a) List the elements $\mathcal{P}_4$, i.e., all the 4-types.
- (b) For each 4-type $P \in \mathcal{P}_4$, give a string of length 4 from the source that has this type.
- (c) For each 4-type $P \in \mathcal{P}_4$, determine the cardinality $|T(P)|$ of the corresponding type class $T(P)$.

Problem 5: Number of types

Show that the exact number of types of a length n sequences of discrete random variables is exactly

$$\binom{n + |\mathcal{X}| - 1}{|\mathcal{X}| - 1}$$

Problem 6: Probabilities of type classes.

Equation (12.23) in the book showed that for every type $P \in \mathcal{P}_n$, we have $P^n(T(P)) \geq P^n(T(\hat{(}P)))$ for all $\hat{P} \in \mathcal{P}_n$.

Show that, for every type $P \in \mathcal{P}_n$, we have that $P^n(T(P)) \geq Q^n(T(P))$ for *every* possible probability distribution Q on the alphabet.