EE595A Submodularity | **University of Washington**
Spring 2011 | **Dept. of Electrical Engineering**

Homework 1: **DUE** April 26th, by 11:45pm Electronically

Prof: J. Bilmes <bilmes@ee.washington.edu> *Tue, April 12th, 2010*

Recall all homework is due electronically via the link https://catalyst.uw.edu/collectit/dropbox/bilmes/14888. Please use PDF if possible.

Problem 1. Min of submodular and constant

We saw in class that if f is a **nondecreasing** submodular set function on S, and q is a real number, then the function f given by

$$f'(U) = \min\{q, f(U)\} \text{ for } U \subseteq S \tag{1}$$

is submodular.

Problem 1(a) Is monotonicity required? If so, show that monotonicity cannot be removed. If not, show that monotonicity is not critical for this property to be true.

Problem 1(b) What about non-increasing (decreasing) functions in min? That is, is submodularity preserved in this case as well when f is monotone non-increasing? Prove or give a counterexample.

Problem 2. matroid

Prove the following theorem.

Theorem 1. *If $\mathcal{V} = (V_i : I \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then for any integer $d \leq |I|$, $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that*

$$f(\cup_{i \in J}\{v_i\}) \geq |J| - d \text{ for all } J \subseteq I \tag{2}$$

if and only if

$$f(V(J)) \geq |J| - d \text{ for all } J \subseteq I \tag{3}$$

You may use any of the theorems from the slides in this class (you may find it useful to look at the slides from lecture 4).

Problem 3. Difference of two submodular functions

Recall, from lecture 1, that in a directed graph, we define $E^+(X, Y) = \{(x, y) \in E(G) : x \in X \setminus Y, y \in Y \setminus X\}$ as the edges from X to Y, $\delta^+(X) = E(X, V \setminus X)$ as the edges edges leaving X, and $\delta^-(X) = E(V \setminus X, X)$ as the edges entering X.

Problem 3(a) Consider the set function $f(A) = |\delta^+(A)| - |\delta^+(V \setminus A)|$. Is this function submodular, supermodular, modular, or neither? Determine which one and prove it.

Problem 4. Concave over modular

Recall from lecture two. Let $m \in \mathbb{R}_+^E$ be any non-negative modular function, and g a concave function over $\mathbb{R}$. Define $f : 2^E \to \mathbb{R}$ as

$$f(A) = g(m(A)) \tag{4}$$

then we stated and partially proved that f is submodular.

Problem 4(a) Prove this theorem in its entirety (the proof in lecture 2 was incomplete).

Problem 4(b) Prove the converse of this theorem, i.e, that if an f formed in such a way is submodular, than it must be that g is a concave function.

Problem 5. Matroids

Recall from lecture three the axioms for a matroid (I1), (I2), and (I3). Recall also from this lecture that we stated that we can replace (I3) with the following condition.

Proposition 2. *In a matroid $M = (E, \mathcal{I})$, for any $U \subseteq E(M)$, any two bases of U have the same size.*

Problem 5(a) In this problem, you are to show that this is true.

Problem 6. Other Min/Max

Let $\{m_i\}_i$ be a finite set of modular functions each $m_i : 2^V \to \mathbb{R}$, and let $\{m_i^+\}_i$ be a finite set of non-negative modular functions each $m_i^+ : 2^V \to \mathbb{R}_+$. Please solve the following problems and please show all work/derivations.

Problem 6(a) Suppose you form function r_m as follows:

$$r_m(A) = \min_i m_i(A) \tag{5}$$

In general, is r_m submodular, supermodular, modular, or neither?

Problem 6(b) Suppose you form function r_m^+ as follows:

$$r_m^+(A) = \min_i m_i^+(A) \tag{6}$$

In general, is r_m^+ submodular, supermodular, modular, or neither?

Problem 6(c) Suppose you form function r_M as follows:

$$r_M(A) = \max_i m_i(A) \tag{7}$$

In general, is r_M submodular, supermodular, modular, or neither?

Problem 6(d) Suppose you form function r_M^+ as follows:

$$r_M^+(A) = \max_i m_i^+(A) \tag{8}$$

In general, is r_M^+ submodular, supermodular, modular, or neither?

Problem 7. Products

Given finite ground set V, and given $w_d \in [0, 1]$ for all $d \in V$, define

$$f(S) = \prod_{d \in S} w_d \tag{9}$$

where $f(\emptyset) = 1$.

Problem 7(a) Is this submodualr, supermodular, modular, or neither?
