EE595A – Submodular functions, their optimization and applications – Spring 2011

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Department of Electrical Engineering
Spring Quarter, 2011
`http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/`

Lecture 10 - May 4th, 2011

Announcements

- On Final projects. **One** single page final project proposals (revision one) are due ~~next~~ this Friday (one week from today) at 6:00pm.
- Again, all submissions must be done electronically, via our drop box. See the link `https://catalyst.uw.edu/collectit/dropbox/bilmes/14888`, or look at the homework on the web page.
- Email me and/or stop by office hours for ideas. The proposals next Friday are non-binding (you can change your mind later) but you should start thinking about project proposals now.
- Ideal proposal would, say, lead to a NIPS paper in June and be related to submodularity.

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- This corresponds to $\min \{r_M(A) + x(E \setminus A) : A \subseteq E\}$ since x is modular and $x(E \setminus A) = x(E) - x(A)$.
- More importantly, $\min \{r_M(A) + x(E \setminus A) : A \subseteq E\}$ a form of submodular function minimization, namely $\min \{\underline{r_M(A) - x(A)} : A \subseteq E\}$ for a submodular function consisting of a difference of matroid rank and modular (so no longer nec. monotone, nor positive).

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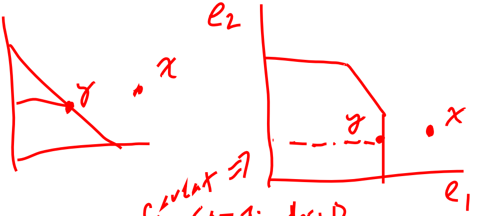
- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*



$y(A) = r_M(A)$

$y(E \setminus A) = x(E \setminus A)$

Ex $A = \{e_1\}$

$E \setminus A = \{e_2\}$

caveat ⇒

Caveat: this is polymatroid but today we are talking about $P_{\{\text{ind. sets}\}}$

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- By the above theorem, the existence of such an A will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, A is minimal in terms of $f(A) \stackrel{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).

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- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of f at A.
- This will also run in polynomial time.

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- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we'll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.
- It has taken us a few lectures to fully develop this algorithm, today we will probably finish it.

Matroid Partition Problem

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- We wish to, if possible, partition E into k blocks, $I_i, i \in \{1, 2, \dots, k\}$ where $I_i \in \mathcal{I}_i$.
- Moreover, we want partition to be lexicographically maximum, that is $|I_1|$ is maximum, $|I_2|$ is maximum given $|I_1|$, and so on.

Matroid Partition Problem

Theorem 2.1 Edmonds 1964

Let M_i be a collection of k matroids as described. Then, a set $I \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \dots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A) \tag{2}$$

where r_i is the rank function of M_i.

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$$|A| \leq k r(A) \;\; \forall A \subseteq E \tag{3}$$
- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as
$$\frac{1}{k}\mathbf{1}(A) \leq r(A) \;\; \forall A \subseteq E \tag{4}$$

Matroid Partition Problem and Submodular Function Minimization

- Recall definition of matroid polytope

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- We also see that this is essentially a special case of submodular function minimization, namely finding A that minimizes $\underline{r(A) - \frac{1}{k}\mathbf{1}(A).}$
- In the general case, we are looking for an A that minimizes
✓ $\sum_i r_i(A) - \mathbf{1}(A)$, and a sum of submodular functions is submodular (in fact, a sum of matroid rank functions is a type of polymatroid rank function Exercise).

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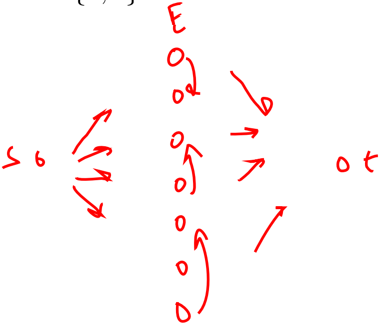

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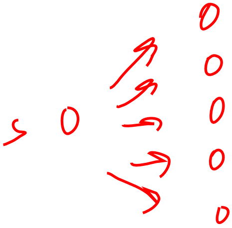

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- Create directed edge (e, t) for all $e \in E$ such that $\exists i \in J$ with $e \notin I_i$ **and** $I_i + e \in \mathcal{I}$. I.e., we add this edge (e, t) if there is some independent set I_i that remains independent if e is added to it.

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- Add directed edge (e, f) for any distinct $e, f \in E$ such that $I_i + e \notin \mathcal{I}$ and $f \in C(I_i, e)$ for some i. That is, we add an edge (e, f) where e directs **to** the elements of a (nec. unique) circuit that is potentially created when e is added to I_i for some i.

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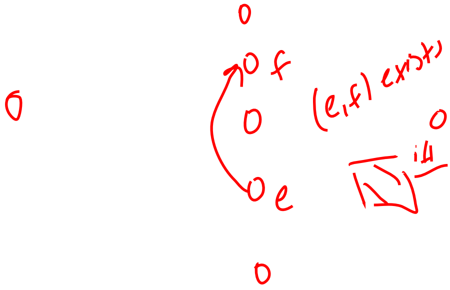
- Therefore, incoming edges to e are either from source s, or from some other node that created a circuit in some I_i.



(e,f) exists iff $\exists\, I_i$ s.t. $I_i + e \notin \mathcal{I}$ and $f \in C(I_i, e)$

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so $S_j + e \in \mathcal{I}$

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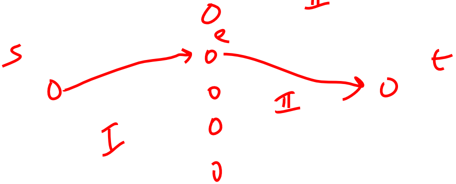
- If the shortest path is $S = (s, e, t)$ then we can add e to some independent set and it is still independent.

I

II



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why shortest, forward ref.

- If the shortest path is $S = (s, e, t)$ then we can add e to some independent set and it is still independent.
- If the shortest path is $S = (s, e, f, t)$ then we can add e to some I_1, create a circuit, but that gets broken when we remove f from that circuit rendering I_1 once again independent, but then there must be some other I_2 that f can be added to w/o making I_2 independent. Thus, the new independent sets are $I_1 + e - f$ and $I_2 + f$, thus we are making progress since overall, e is added.

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  3. add f_1 to some I_2, thus making a circuit C_2 due to edge (f_1, f_2).

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  3. add f_1 to some I_2, thus making a circuit C_2 due to edge (f_1, f_2).
  4. subtract f_2 from I_2, eliminating the circuit C_2.
  5. add f_2 to some I_3, not making a circuit due to edge (f_2, t).

  thus making progress.

Matroid Partition - Flow solution when $M = M_i, \forall i$

- Note that $I_1 \neq I_2$ since the edge (f, t) meant that we originally had $f \notin I_2$.
- If the shortest path is $S = (s, e, f_1, f_2, t)$ then we can:
  1. add e to some I_1, thus making a circuit C_1 due to edge (e, f_1).
  2. subtract f_1 from I_1, eliminating the circuit C_1.
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  thus making progress.
- Here, $I_1 \neq I_2$, and $I_2 \neq I_3$, but could have $I_1 = I_3$ Exercise:

Flow solution theorem

Thus, we have outlined the proof of one direction in the following theorem. When all matroids are the same $\forall i, M_i = M$ for some matroid, we have:

Theorem 3.1

There is an (s, t) path in the aforementioned graph iff the set of independent sets $(I_i : i \in J)$ can be grown by one element and still be a partition of some subset of E.

The other direction can be shown as a consequence of Theorem 2.1. Exercise

Problem To Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathcal{R}_+^E$;
- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*
- By the above theorem, the existence of such an A will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, A is minimal in terms of $f(A) \stackrel{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).
- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of f at A.
- This will also run in polynomial time.

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- We gradually build up y by adding new independent sets (and augmenting J), adding to the existing independent sets, and adjusting coefficients.
- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we'll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.

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- The algorithm starts with $y = 0$, $J = \{0\}$, $I_0 = \emptyset$, and $\lambda_0 = 1$.

Augmenting path theorem

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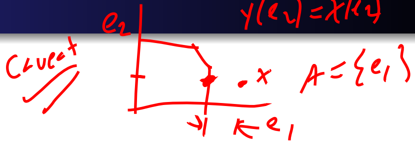

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If there is a directed path from s to t in G, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

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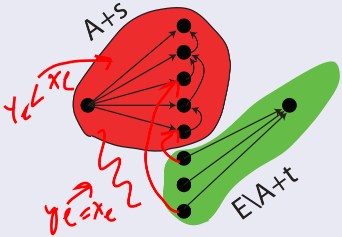

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9. we have that $I_i + e \in \mathcal{I}$, implying that (e, t) is an edge in G (impossible since $(s, e) \in G$, so can't also have $(e, t) \in G$ since no s, t path in G).

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⑩ alternatively, $I_i + e \notin \mathcal{I}$, so circuit $C(I_i, e)$ exists which can not be contained in A. *(we needed in (7) that $(I_i \cap A) + e$ is independent, and if the circuit was fully in A then this independence consequence would not hold).*

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11. Thus, in this case, there exists $f \in I_i \setminus A$ such that $f \in C(I_i, e)$, and also edge $(e, f) \in G$. But this also can't happen since this would be a zero-edge cut crossing edge. Thus, (6) is true.

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⑫ Therefore, since $y = \sum_{i \in J} \lambda_i \mathbf{1}_{I_i}$, we have:

$$y(A) = \sum_{a \in A} y_a = \sum_{i \in J} \lambda_i \mathbf{1}_{I_i}(A) \tag{6}$$

$$= \sum_{i \in J} \lambda_i |I_i \cap A| = \sum_{i \in J} \lambda_i r(A) = r(A) \qquad \text{as required.} \tag{7}$$

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Proof of Thm 4.1.

⑬ Conversely, suppose there is a directed s, t path in G, and it is given by sequence $S = (e_1, e_2, \dots, e_m)$ of distinct elements. $e_i \in E$

$S = s = e_0, e_1, \dots, e_m, e_{m+1} = t$

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14. Then there is a mapping $i : [m] \rightarrow J$ (where the $i(j)$ are not distinct), satisfying

$\leftarrow \subset \{1, .., m\}$

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$$(e_j, e_{j+1}) \in G \Rightarrow e_{j+1} \in C(I_{i(j)}, e_j) \text{ for } 1 \leq j \leq m-1; \tag{10}$$

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If there is a directed path from s to t in G, then there exists $y' \in P$ *with* $y \leq y' \leq x$, *with* $y'(E) > y(E)$. *If there is no such path, then there exists a set* $A \subseteq E$ *s.t.* $y(A) = r(A)$ *and* $y(E \setminus A) = x(E \setminus A)$.

Proof of Thm 4.1.

⑬ Conversely, suppose there is a directed s, t path in G, and it is given by ordered sequence $S = (e_1, e_2, \dots, e_m)$ of distinct elements.

⑭ Then there is a mapping $i : [m] \to J$ (where the $i(j)$ are not distinct), satisfying — distinct w.r.t. S

$$(s, e_1) \in G \Rightarrow x_{e_1} > y_{e_1}; \tag{8}$$

$$(e_1, e_2) \in G \Rightarrow e_2 \in C(I_{i(1)}, e_1); \tag{9}$$

$$(e_j, e_{j+1}) \in G \Rightarrow e_{j+1} \in C(I_{i(j)}, e_j) \text{ for } 1 \leq j \leq m-1; \tag{10}$$

$$(e_m, t) \in G \Rightarrow I_{i(m)} + e_m \in \mathcal{I}. \tag{11}$$

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Proof of Thm 4.1.

$i_s(j)$

⑮ Now, for $i \in J$, define $k_i = |\{j : i = i(j)\}| = \cup_{j \in J} 1_{i=i(j)}$ be the number of times that the i'th independent set I_i is used in the mapping $i : [m] \to J$.

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16. Let $\delta > 0$ be small.
17. Update λ: For each $i \in J$, set $\lambda_i \leftarrow \lambda_i - k_i \delta$. So now, $\sum_i \lambda_i = 1 - m\delta$. since $\sum_i k_i = m$

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part of the alg.

⑱ Next, we add e_1 to $\cup_i I_i$, and distribute amongst them to remove any circuits, as follows.

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for $j \in 1 \ldots m-1$ do
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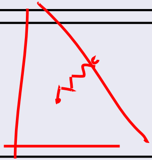

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so [illegible] is ok since not yet saturated before.

㉑ The theorem is proven. $y' \in P$

Augmenting path theorem consequences

Corollary 4.2

For any $x \in \mathbb{R}_+^E$, we have

$$\max\left(y(E) : y \le x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{12}$$

Note: this was not used in the theorem above, rather it is a consequence!

Proof.

1. First, any $y \in \underset{\text{ind}}{P}$ with $y \le x$, and any $A \subset E$, we have

$$y(E) = y(A) + y(E \setminus A) \underline{\le} r(A) + x(E \setminus A) \tag{13}$$

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5. Then, there is a set A such that $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$, or that $y(E) = r(A) + x(E \setminus A)$, thus demonstrating equality.

□

Augmenting path theorem consequences

Corollary 4.3

Given matroid M, we have

$$P_{ind.\ set} = P_r \tag{14}$$

We even get this a consequence!

Proof.

- We saw before (in lecture 7) that this follows from corollary 4.2 (which we encountered in lecture 7).

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Proof.

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- Therefore, the equivalence follows indirectly just from Theorem 4.1!!

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- Key in this is to: 1) scan nodes in the order that they are labeled, and 2) label nodes (from a node being scanned) in an order consistent with some fixed total order on all vertices.
- While 1) ensures that the path has as few edges as possible (proven in Edmonds/Karp), 2) results in a lexicographically minimum order. Both together are called a *consistent breadth-first search*, or CBFS.

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    1. In every path used for augmentation, there is one critical edge, that becomes unavailable for use immediately after the augmentation.
    2. The only way an edge becomes available for use in an augmenting path is by being used in the opposite direction in the previous augmentation.
- On our current context, we have results quite similar to this that guarantee that the number of augmentations is polynomially bounded, yielding our next theorem.

Bounding the number of augmenting paths

- Consider the algorithm implied by Theorem 4.1 as producing one augmentation, and let G_i refer to the digraph at outer iteration i. Then we have

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Let $G_0, G_1, \dots, G_k$ *be a sequence of digraphs, each having vertex set* $E \cup \{s, t\}$*, and correspond to such graphs each one running the algorithm implied by theorem 4.1 Assume fixed total order of* $E \cup \{s\}$*. Let* Q_i *denote the CBFS path in* G_i*, for* $0 \leq i < k$*. If it is the case that, for* $0 \leq i < k$*:*

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Then we have that the number of augmentations has bound $k \leq |E|^3$.

Bounding the number of augmenting paths

Theorem 4.5

It is possible to construct an augmentation scheme such that each augmenting path is done in accordance to Theorem 4.4. Each such augmentation is CBFS, and is called a "grand" augmentation, and is maximal in a certain way. This achieves the $O(n^3)$ time, in the number of augmentations, mentioned above.

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- Of course, the cost of each augmentation might be expensive. For matric matroids, each would be $\underline{O(r^2n^5)}$ where r is the number of rows of the matrix, leading to $\underbrace{O(r^2n^8)}$ algorithm.

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- On the other hand, this algorithm has some intriguing properties.

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  3. This works only for the difference between r and x, but we'd like an algorithm that works for any arbitrary submodular function f, even non-monotone and/or non-non-increasing/decreasing.
- It turns out that (2) and (3) is easy to deal with, but (1) took another 16 years to solve (and perhaps can still be seen as unsolved, w.r.t. wanting a scalable algorithm).

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- Then $f(\emptyset) = 0$, so f is normalized.

Towards SFM

- First, given any submodular function g, construct modular function $m : E \to \mathbb{R}$ such that $m(e) = g(E \setminus \{e\}) - g(E)$.
- Then construct a new function $f : 2^E \mathbb{R}_+$ as

$$f(A) = g(A) + m(A) - g(\emptyset) \tag{15}$$

- Then $f(\emptyset) = 0$, so f is normalized.
- Also, f is monotone non-decreasing and submodular. It is submodular since sum of submodular and modular. Monotone non-decreasing follows since

$$\begin{aligned} f(B+v) - f(B) &= g(B+v) - g(B) + m(v) && (16) \\ &= g(B+v) - g(B) + g(E-v) - g(E) && (17) \\ &\geq 0 && (18) \end{aligned}$$

since, by submodularity, $g(B+v) - g(B) \geq g(E-v) - g(E)$.

Towards SFM

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- Is $m \in \mathbb{R}_+^E$?
- No, but for any e such that $m(e) < 0$ can't be a minimizer of $f - m$ since, assuming that A minimizes $f(A) - m(A)$ and $e \in A$ is such that $m(e) < 0$, then we have that $f(A') - m(A') < f(A) - m(A)$ where $A' = A \setminus \{e\}$.

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- Therefore, SFM is as "easy" as moving from matroid rank functions to not-necessarily-integral polymatroidal functions.

Scratch Paper

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Sources for Today's Lecture

- Jack Edmonds, "Matroid Partition", 1968.
- W. Cunningham, "Testing Membership in Matroid Polyhedra", 1984
- E. Lawler, "Matroid Intersection Algorithms", 1975.
- L. Schrijver, "Combinatorial Optimization", 2003.
- Krogdahl, "A Combinatorial Base for some Optimal Matroid Intersection Algorithms", 1974.