

EE595A – Submodular functions, their optimization and applications – Spring 2011

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering
Spring Quarter, 2011

http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 13 - May 13th, 2011

Announcements

- On Final projects. **One** single page final project updates due next Wednesday, 5/18 at 11:45pm.
- Again, all submissions must be done electronically, via our drop box. See the link
<https://catalyst.uw.edu/collectit/dropbox/bilmes/14888>,
or look at the homework on the web page.
- Homework 2 is due next Friday night at 11:45pm. All things in lectures marked “exercise”

Class Road Map

We need to find one makeup lectures this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L11 (5/6): On SFM, polymatroid member & greedy, Lovász ext.
- L12 (5/11): Lovász ext. + polymatroid props.
- L13 (5/13): More polymatroids, start lattices
- L14 (5/18):
- L15 (5/20):
- L16 (5/25):
- L17 (5/27):
- L18 (6/1):
- L19 (6/3):
- L20: (6/?): (need to find time/date/place).

A Lecture and a Course by Jack Edmonds, Rome, May 23-27, 2011

POLYMATROIDS

The talk will sketch an introduction to P , NP , $coNP$, LP duality, matroids, and some other foundations of combinatorial optimization theory. A predicate, $p(x)$, is a statement with variable input x . It is said to be in NP when, for any x such that $p(x)$ is true, there is, relative to the bit-size of x , an easy proof that $p(x)$ is true. It is said to be in $coNP$ when $\neg p(x)$ is in NP . It is said to be in P when there is an easy (i.e., polynomially bounded time) algorithm for deciding whether or not $p(x)$ is true. Of course P implies NP and $coNP$. Fifty years ago I speculated the converse. Polymatroids are a linear programming construction of abstract matroids. We use them to describe large classes of concrete predicates (i.e., “problems”) which turn out to be in NP , in $coNP$, and indeed in P . Failures in trying to place the NP “traveling salesman predicate” in $coNP$, and successes in placing some closely related polymatroidal predicates in both NP and $coNP$ and then in P , prompted me to conjecture that (1) the NP traveling salesman predicate is not in P , and (2) all predicates in both NP and $coNP$ are in P . The conjectures have become popular, and are both used as practical axioms. I might as well conjecture that the conjectures have no proofs.

Monday, May 23 2011, 11:30

A Lecture and a Course by Jack Edmonds, Rome, May 23-27, 2011

POLYMATROIDS ETCETERA: ALGORITHMS AND PRETTY THEOREMS

A variety of combinatorial existence theorems will be proved by algorithms which tell how to find an instance of what is asserted to exist. Another main purpose will be to introduce polyhedral combinatorics, which uses systems of linear equations to obtain algorithms and theorems. Emphasis will be on matroids and polymatroids with a variety of applications, especially to tree systems and branching systems in networks.

Tuesday-Friday, May 24-27 2011, 10:30

A Lecture and a Course by Jack Edmonds, Rome, May 23-27, 2011

Both may be seen at <http://www.iasi.cnr.it/jack/>



A LECTURE AND A COURSE BY JACK EDMONDS

ROME, MAY 23-27, 2011

ISTITUTO DI ANALISI DEI SISTEMI ED INFORMATICA - CONSIGLIO NAZIONALE DELLE RICERCHE

Conference Room, viale Manzoni 30 - Roma

Recent reprints of famous papers:

- Matroid Partition**
 in the book

- Submodular Functions, Matroids, and Certain Polyhedra**
 • Theoretical Improvements in Algorithms for Network Flow Problems (with Z. S. Gal)
 • Matching: A Well-Invited Class of Integer Linear Programs (with E. L. Johnson)

in the book dedicated to Jack



The lecture:

POLYMATROIDS

The talk will start with an introduction to P , NP , $coNP$, LP duality, matroids, and some other foundations of combinatorial optimization theory.

A predicate $p(x)$ is a statement with variable x and it is said to be in NP when, for any x such that $p(x)$ is true, there is relative to the bit-size of x , an easy proof that $p(x)$ is true. It is said to be in $coNP$ when $\neg p(x)$ is in NP . It is said to be in P when there is an easy (i.e., polynomially bounded time) algorithm for deciding whether or not $p(x)$ is true. Of course P implies NP and $coNP$.

NP soon gets upgraded to the universe.

Poly matroids are a linear programming construction of abstract matroids. We use them to describe large classes of concrete problems (i.e., "problems") which can be in NP , in $coNP$, and indeed in P .

Failure in trying to place the NP "hardness" predicate in $coNP$ and, conversely, to place some clearly related polynomial predicate in both NP and $coNP$ and then in P , prompted me to conjecture that:

- (1) the NP "hardness" predicate is not in P , and
- (2) all predicates in both NP and $coNP$ are in P .

The conjectures have become popular, and are both used in practical areas. I might as well conjecture that the conjectures have no proof.

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[Download the audio](#)

The course:

POLYMATROIDS ETCETERA: ALGORITHMS AND PRETTY THEOREMS

A variety of combinatorial existence theorems will be proved by algorithms which will help to find an instance of what is asserted to exist. Another main purpose will be to introduce polynomial combinatorics, which consists of linear equations in whose algorithms and theorems. Emphasis will be on matroids and polymatroids with a variety of algorithms, especially to two systems and bounding systems in networks.

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If you like to receive reminders or notification of possible schedule changes, please drop your e-mail address into the box below:

from the book "Combinatorial Optimization", by Len Schrijver

"The class of problems which are respectively known and not known to have good algorithms are of great theoretical interest. [...] I conjecture that there is no good algorithm for the remaining unknown problems. My reasons are the same as for my mathematical conjectures: (1) it is a legitimate mathematical possibility, and (2) it has not been" -- Jack Edmonds, 1988

from the book "Computational Complexity", by Christos Papadimitriou

"Jack Edmonds has been one of the creators of the field of combinatorial optimization and polynomial combinatorics. His 1983 paper "Paths, Trees and Flowers" was one of the first papers to suggest the possibility of establishing a mathematical theory of NP-combinatorial optimization."

from the citation of the John Von Neumann Theory Prize, 1981

Gallery:



Jack Edmonds is awarded an Honorary Doctorate at the University of Cordoba, Decree and is organized by Queen Margarita II.



Assisi 2003



Assisi 2008



An extension of f

- For any f (even not submodular), we can define an extension in this way, with

$$\tilde{f}(w) = \sum_{i=1}^m \lambda_i f(U_i) \quad (1)$$

with the U_i 's and sorted order of w defined as above, so that

$$w = \sum_{i=1}^m \lambda_i \mathbf{1}_{U_i}$$

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A function $f : 2^E \rightarrow \mathbb{R}$ is submodular iff its Lovász extension \tilde{f} of f is convex.

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- Perhaps we could call this the Edmonds-Lovász-Choquet extension?

Choquet integral

Definition 2.2

Let f be any capacity on E and $w \in \mathbb{R}_+^E$. The **Choquet integral** (1954) of w w.r.t. f is defined by

$$C_f(w) = \sum_{i=1}^m (w_{e_i} - w_{e_{i+1}}) f(U_i) \quad (2)$$

where in the sum, we have sorted and renamed the elements of E so that $w_{e_1} \geq w_{e_2} \geq \dots \geq w_{e_m} \geq w_{e_{m+1}} = 0$, and where $U_i = \{e_1, e_2, \dots, e_i\}$.

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Definition 2.3

Given $w \in \mathbb{R}_+^E$, the Lovász extension (equivalently Choquet integral) may be defined as follows:

$$\tilde{f}(w) \stackrel{\text{def}}{=} \int_0^\infty F(\alpha) d\alpha \quad (3)$$

where the function F is defined as before.

Lovász extension

- For a given $w \in [0, 1]^m$, it is easy to see that we can also define the Lovász extension as

$$\tilde{f}(w) = \mathbb{E}[f(e \in E : w(e_i) > \alpha)] \quad (4)$$

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- The convexity of the Lovász extension, the ease of minimizing convex functions, and the fact that we can recover f from \tilde{f} via $f(A) = \tilde{f}(\mathbf{1}_A)$ corresponds to why SFM is possible in polynomial time (which was first shown by Grötschel, Lovász, and Schrijver in 1988 as part of their Ellipsoid method).

Choquet integral and aggregation

- Given the following form of aggregation

$$\text{AG}(x) \stackrel{\text{def}}{=} \sum_{A: \mathbf{1}_A \in \mathcal{V}(x)} \left(\alpha_0^x(A) + \sum_{i=1}^m \alpha_i^x(A) x_i \right) \text{AG}(\mathbf{1}_A) \quad (5)$$

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- We can define a canonical triangulation of the hypercube in terms of permutations of the coordinates. I.e., given some permutation σ , define

$$\text{conv}(\mathcal{V}_\sigma) = \{x \in [0, 1]^n \mid x_{\sigma(1)} \geq x_{\sigma(2)} \geq \dots \geq x_{\sigma(m)}\} \quad (6)$$

Then these $m!$ blocks of the partition are called the **canonical partitions** of the hypercube. In this case, we have:

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Proposition 2.4

The above linear interpolation using the canonical partition yields the Lovász extension.

Polymatroid extreme points

Theorem 2.5

For a given ordering $E = (e_1, \dots, e_m)$ of E and a given E_i and x generated by E_i using the greedy procedure, then x is an extreme point of P_f

Polymatroid extreme points

- The greedy algorithm does more than solve $\max(w^T x : x \in P_f)$. We can use it to generate vertices of polymatroidal polytopes.

Polymatroid extreme points

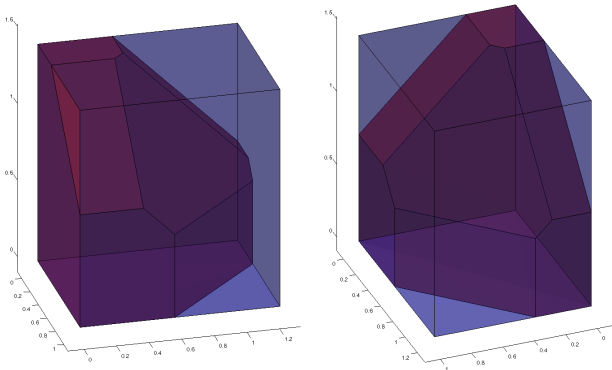
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$$B_f = P_f \cap \left\{ x \in \mathbb{R}_+^E : x(E) = f(E) \right\} \quad (7)$$

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- Also, intuitively, we can continue advancing along the skeletal edges of the polytope to reach any other vertex, given the appropriate ordering. If we advance in all dimensions, we'll reach a vertex in B_f , and if we advance only in some dimensions, we'll reach a vertex in $P_f \setminus B_f$.

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- We formalize this next:

Polymatroid extreme points

- Given any arbitrary order of $E = (e_1, e_2, \dots, e_m)$, define $E_i = (e_1, e_2, \dots, e_i)$.

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- Given any arbitrary order of $E = (e_1, e_2, \dots, e_m)$, define $E_i = (e_1, e_2, \dots, e_i)$.

- A vector x is generated by E_i using the greedy procedure as follows

$$x(e_1) = f(E_1) \quad (8)$$

$$x(e_j) = f(E_j) - f(E_{j-1}) \text{ for } 2 \leq j \leq i \quad (9)$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \quad (10)$$

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$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{10}$$
- An **extreme point** of P_f is a point that is not a convex combination of two other distinct points in P_f . Equivalently, an extreme point corresponds to setting certain inequalities in the specification of P_f to be equalities, so that there is a unique single point solution.

Polymatroid extreme points

Theorem 3.1

For a given ordering $E = (e_1, \dots, e_m)$ of E and a given E_i and x generated by E_i using the greedy procedure, then x is an extreme point of P_f

Polymatroid extreme points

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Proof.

- We already saw that $x \in P_f$ (in Lecture 11, proof of Theorem 4.2).



Polymatroid extreme points

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Proof.

- We already saw that $x \in P_f$ (in Lecture 11, proof of Theorem 4.2).
- To show that x is an extreme point of P_f , note that it is the unique solution of the following system of equations

$$x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \quad (11)$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \quad (12)$$



Polymatroid extreme points

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- $x(E_3) = x(e_1) + x(e_2) + x(e_3) = f(e_1, e_2, e_3)$ so
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- And so on ...
- Also, since $x \in P_f$, for each i , we see that,

$$x(E_i) = f(E_i) \tag{13}$$

$$x(A) \leq f(A), \forall A \subseteq E \tag{14}$$

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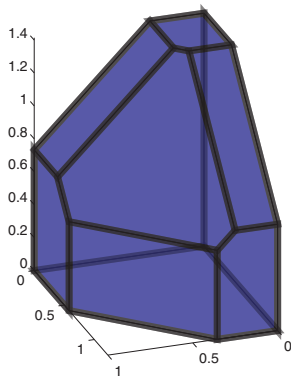
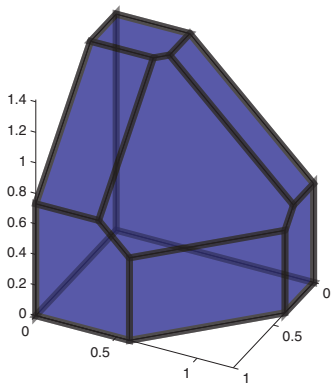
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- Thus, the greedy procedure provides a modular function lower bound on f that is tight on all points E_i in the order.

Polymatroid extreme points

some examples



Polymatroid extreme points

- Moreover, we have

Corollary 3.2

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A))$, then x is generated using greedy by some ordering of B .

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- Note, $\text{cl}(x) = \cup(A : x(A) = f(A))$ is the closure of x (recall that sets A such that $x(A) = f(A)$ are called **tight**, and such sets are closed under union and intersection, see Lecture 7, in proof of Theorem 4.3, starting Eq. 50).

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- Thus, $\text{cl}(x)$ is a tight set.
- Also, $\text{supp}(x) = \{e \in E : x(e) \neq 0\}$ is called the support of x .

Polymatroid extreme points

- Moreover, we have

Corollary 3.2

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A))$, then x is generated using greedy by some ordering of B .

- Note, $\text{cl}(x) = \cup(A : x(A) = f(A))$ is the closure of x (recall that sets A such that $x(A) = f(A)$ are called **tight**, and such sets are closed under union and intersection, see Lecture 7, in proof of Theorem 4.3, starting Eq. 50).
- Thus, $\text{cl}(x)$ is a tight set.
- Also, $\text{supp}(x) = \{e \in E : x(e) \neq 0\}$ is called the support of x .
- For arbitrary x , $\text{supp}(x)$ is not tight, but for an extreme point, $\text{supp}(x)$ is.

Polymatroid Closure/Sat

- Recall closure from Lecture 3: Given $A \subseteq E$, the **closure** or **span** of A , is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$ where r is matroid rank.

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- For a given $x \in P_f$, we can define this family

$$\mathcal{D}(x) = \{A : A \subseteq E, x(A) = f(A)\} \quad (15)$$

Polymatroid Closure/Sat

- Now given:

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- Then since $f(A) = f(B) \leq f(A \cap B)$ and $f(A) = f(B) \leq f(A \cup B)$, and by submodularity

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (18)$$
 we must have $f(A) = f(B) = f(A \cup B) = f(A \cap B)$.

Polymatroid Closure/Sat

- Matroid closure can be generalized (also called the polymatroid closure or **saturation function**) as unique maximal element in $\mathcal{D}(x)$. That is, for some $x \in P_f$, we define:

$$\text{cl}(x) \stackrel{\text{def}}{=} \text{sat}(x) \stackrel{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x)\} \quad (19)$$

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- This generalizes matroid closure in the following way (see next slide):

Polymatroid Closure/Sat

- Consider matroid $(E, \mathcal{I}) = (E, r)$, some $I \in \mathcal{I}$. Then $\mathbf{1}_I \in P_r$ and $\mathcal{D}(\mathbf{1}_I) = \{A : \mathbf{1}_I(A) = r(A)\}$, and

$$\text{sat}(\mathbf{1}_I) = \bigcup \{A : A \subseteq E, A \in \mathcal{D}(\mathbf{1}_I)\} \quad (22)$$

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- Given $A \in \mathcal{D}(\mathbf{1}_I)$, and $b \in \text{span}(A) \setminus A$, then

$$r(A + b) = r(A) = |I \cap A| = r(I \cap A) \quad (25)$$

so $b \notin I \setminus A$ (since otherwise it would increase the rank) and so $|I \cap (A + b)| = |I \cap A|$, meaning $A + b \in \mathcal{D}(\mathbf{1}_I)$.

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- Thus, $\text{sat}(\mathbf{1}_I) \supseteq \text{span}(I)$.
- Now, consider $b \in \text{sat}(\mathbf{1}_I) \setminus I$. Choose any $A \in \mathcal{D}(\mathbf{1}_I)$ with $b \in A$. Then $r(A) = r((I \cap A) \cup (A \setminus I)) = r(I \cap A) = r((I \cap A) + b)$.

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- Let $I_B \in \mathcal{I}$ be such that $I_B \subseteq B \subseteq \text{span}(I_B)$, so $r(B) = r(I_B)$. Then, make the definition:

$$\text{sat}(\mathbf{1}_B) = \text{sat}(\mathbf{1}_{I_B}) \quad (27)$$

In which case, we also get $\text{sat}(\mathbf{1}_B) = \text{span}(B)$.

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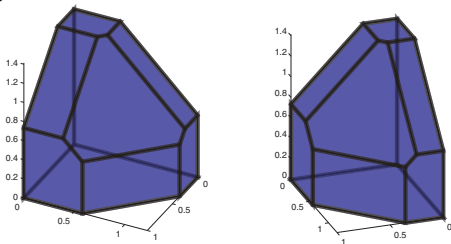
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- For any extremal point x , we have $\text{supp}(x)$ is tight since extremal points are defined as a system of equalities of the form $x(E_i) = f(E_i)$ as we saw earlier in lecture.



Dependence Function

- Tight sets can be restricted to contain a particular element. Given $x \in P_f$, and $e \in \text{sat}(x)$, define

$$\mathcal{D}(x, e) = \{A : e \in A \subseteq E, x(A) = f(A)\} \quad (29)$$

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- Therefore, we can define a unique minimal element of $\mathcal{D}(x, e)$ denoted as follows:

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases} \quad (31)$$

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- I.e., $\text{dep}(x, e)$ is the minimal element in $\mathcal{D}(x)$ that contains e (the minimal x -tight set containing e).

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- Now, let $(E, \mathcal{I}) = (E, r)$ be a matroid, and let $I \in \mathcal{I}$ giving $\mathbf{1}_I \in P_r$. Let $e \in \text{sat}(\mathbf{1}_I) = \text{span}(I) = \text{closure}(I)$.

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- Then $\text{dep}(\mathbf{1}_I, e)$ must be a circuit since if it included more than a circuit, it would not be minimal in this sense.

Dependence Function

- Therefore, when $e \in \text{sat}(\mathbf{1}_I) \setminus I$, then $\text{dep}(\mathbf{1}_I, e) = C(I, e)$ where $C(I, e)$ is the unique circuit contained in $I + e$ in a matroid (the **fundamental circuit** of e in the independent set I we encountered in Lecture 7).

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- In this case, for such an e , we have $\text{dep}(\mathbf{1}_I, e) = \{e\}$ since all such sets $A \ni e$ with $|I \cap A| = r(A)$ contain e , but in this case no cycle is created.

Dependence Function

- Therefore, when $e \in \text{sat}(\mathbf{1}_I) \setminus I$, then $\text{dep}(\mathbf{1}_I, e) = C(I, e)$ where $C(I, e)$ is the unique circuit contained in $I + e$ in a matroid (the **fundamental circuit** of e in the independent set I we encountered in Lecture 7).
- Now, if $e \in \text{sat}(\mathbf{1}_I) \cap I$ with $I \in \mathcal{I}$, we said that $C(I, e)$ was undefined (since no circuit is created in this case).
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- We are thus free to take subsets of I as A , all of which must contain e , but all of which have rank equal to size.
- Also note: in general for $x \in P_f$ and $e \in \text{sat}(x)$, we have $\text{dep}(x, e)$ is tight by definition.

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 - For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated (x -tight) set w.r.t. x . I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$
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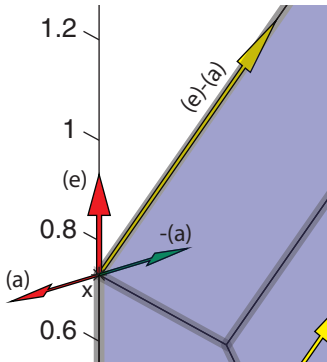
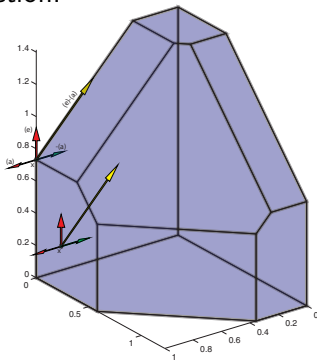
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- Analogously, for $e \in \text{sat}(x)$, any $x + \alpha \mathbf{1}_e \notin P_f$ for $\alpha > 0$.
- But, analogous to the circuit case, is there an exchange property for $\text{dep}(x, e)$? Could move in this direction if we simultaneously move in another direction?

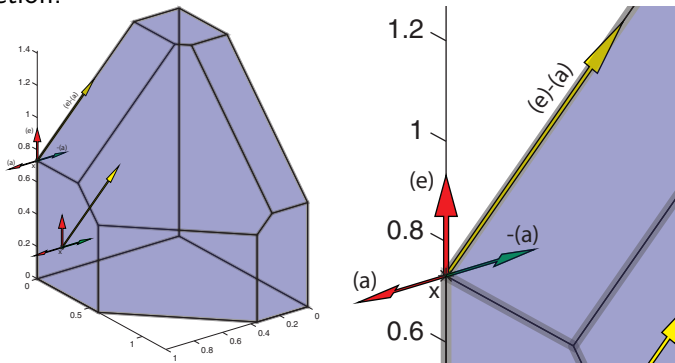
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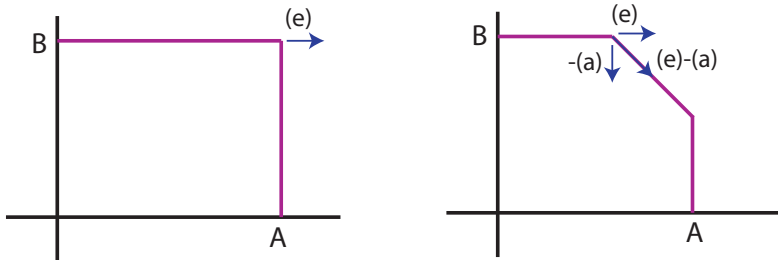
- This suggests, for $e \in \text{sat}(x)$, that

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- Viewable in 2D, we have for $A, B \subseteq E$, $A \cap B = \emptyset$:



Left: $A \cap \text{dep}(x, e) = \emptyset$, and we can't move further in (e) direction by moving in any negative $a \in A$ direction.

Right: $A \subseteq \text{dep}(x, e)$, and we can move further in (e) direction by moving in some $a \in A$ negative direction.

Extreme points by greedy

Recall earlier theorem

Corollary 3.3

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) \subseteq B \subseteq \text{sat}(x)$, then x is generated using greedy by some ordering of B .

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- We then choose c such that x is the desired vertex. This is then achieved by greedy.

Outline

- We're next going to study lattices and submodular functions.
- In doing so, we'll better be able to understand certain properties of polymatroidal extreme points and ultimately SFM.

Scratch Paper

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Sources for Today's Lecture

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