EE595A – Submodular functions, their optimization and applications – Spring 2011

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering

Spring Quarter, 2011
http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 13 - May 13th, 2011

Announcements

- On Final projects. **One** single page final project updates due next Wednesday, 5/18 at 5:00pm.
- Again, all submissions must be done electronically, via our drop box. See the link `https://catalyst.uw.edu/collectit/dropbox/bilmes/14888`, or look at the homework on the web page.
- HW2 is due next friday. Look for "exercise" in slides.

Class Road Map

We need to find one makeup lectures this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L11 (5/6): On SFM, polymatroid member & greedy, Lovász ext.
- L12 (5/11): Lovász ext. + polymatroid props.
- L13 (5/13): More polymatroids, start lattices
- L14 (5/18):
- L15 (5/20):
- L16 (5/25):
- L17 (5/27):
- L18 (6/1):
- L19 (6/3):
- L20: (6/?): (need to find time/date/place).

A Lecture and a Course by Jack Edmonds, Rome, May 23-27, 2011

POLYMATROIDS

The talk will sketch an introduction to P, NP, coNP, LP duality, matroids, and some other foundations of combinatorial optimization theory. A predicate, p(x), is a statement with variable input x. It is said to be in NP when, for any x such that p(x) is true, there is, relative to the bit-size of x, an easy proof that p(x) is true. It is said to be in coNP when ¬p(x) is in NP. It is said to be in P when there is an easy (i.e., polynomially bounded time) algorithm for deciding whether or not p(x) is true. Of course P implies NP and coNP. Fifty years ago I speculated the converse. Polymatroids are a linear programming construction of abstract matroids. We use them to describe large classes of concrete predicates (i.e., "problems") which turn out to be in NP, in coNP, and indeed in P. Failures in trying to place the NP "traveling salesman predicate" in coNP, and successes in placing some closely related polymatroidal predicates in both NP and coNP and then in P, prompted me to conjecture that (1) the NP traveling salesman predicate is not in P, and (2) all predicates in both NP and coNP are in P. The conjectures have become popular, and are both used as practical axioms. I might as well conjecture that the conjectures have no proofs.

Monday, May 23 2011, 11:30

A Lecture and a Course by Jack Edmonds, Rome, May 23-27, 2011

POLYMATROIDS ETCETERA: ALGORITHMS AND PRETTY THEOREMS
A variety of combinatorial existence theorems will be proved by algorithms which tell how to find an instance of what is asserted to exist. Another main purpose will be to introduce polyhedral combinatorics, which uses systems of linear equations to obtain algorithms and theorems. Emphasis will be on matroids and polymatroids with a variety of applications, especially to tree systems and branching systems in networks.
Tuesday-Friday, May 24-27 2011, 10:30

A Lecture and a Course by Jack Edmonds, Rome, May 23-27, 2011

Both may be seen at http://www.iasi.cnr.it/jack/

A LECTURE AND A COURSE BY JACK EDMONDS
ROME, MAY 23-27, 2011

ISTITUTO DI ANALISI DEI SISTEMI ED INFORMATICA - CONSIGLIO NAZIONALE DELLE RICERCHE
Conference Room, viale Manzoni 30 - Roma

Recent reprints of famous papers:

- Matroid Partition

in the book

50 Years of Integer Programming 1958–2008

- Submodular Functions, Matroids, and Certain Polyhedra
- Theoretical Improvements in Algorithmic Efficiency for Network Flow Problems (with R.M. Karp)
- Matching: A Well-Solved Class of Integer Linear Programs (with E.L. Johnson)

in the book dedicated to Jack

Combinatorial Optimization – Eureka, You Shrink!

The lecture:

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Download the poster

The course:

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If you like to receive reminders or notification of possible schedule changes, please drop your e-mail address into the box below

and press this button

"Pioneered by the work of Jack Edmonds, polyhedral combinatorics has proved to be a most powerful, coherent and unifying tool throughout combinatorial optimization. ..."
"Edmonds conjectured that there is no polynomial-time algorithm for the traveling salesman problem. In language that was developed later, this is equivalent to NP≠P."
from the book 'Combinatorial Optimization', by Lex Schrijver

"The classes of problems which are respectively known and not known to have good algorithms are of great theoretical interest. [...] I conjecture that there is no good algorithm for the traveling salesman problem. My reasons are the same as for any mathematical conjecture: (1) It is a legitimate mathematical possibility, and (2) I do not know." – Jack Edmonds, 1966
from the book 'Computational Complexity', by Christos Papadimitriou

"Jack Edmonds has been one of the creators of the field of combinatorial optimization and polyhedral combinatorics. His 1965 paper 'Paths, Trees and Flowers' was one of the first papers to suggest the possibility of establishing a mathematical theory of efficient combinatorial algorithms"
from the citation of the John Von Neumann Theory Prize, 1985

Gallery:

Jack à la Sorbonne

Jack Edmonds is awarded an Honorary Doctorate at the University of Southern Denmark and is congratulated by Queen Margrethe II.

Aussois 2003

Aussois 2008

An extension of f

- For any f (even not submodular), we can define an extension in this way, with

$$\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(U_i) \quad (1)$$

with the U_i's and sorted order of w defined as above, so that $w = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{U_i}$

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Theorem 2.1

A function $f : 2^E \to \mathbb{R}$ is submodular iff its Lovász extension $\tilde{f}$ of f is convex.

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- Perhaps we could call this the Edmonds-Lovász-Choquet extension?

Choquet integral

Definition 2.2

Let f be any capacity on E and $w \in \mathbb{R}_+^E$. The Choquet integral (1954) of w w.r.t. f is defined by

$$C_f(w) = \sum_{i=1}^{m} (w_{e_i} - w_{e_{i+1}}) f(U_i) \tag{2}$$

where in the sum, we have sorted and renamed the elements of E so that $w_{e_1} \geq w_{e_2} \geq \cdots \geq w_{e_m} \geq w_{e_{m+1}} = 0$, and where $U_i = \{e_1, e_2, \ldots, e_i\}$.

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Definition 2.3

Given $w \in \mathbb{R}_+^E$, the Lovász extension (equivalently Choquet integral) may be defined as follows:

$$\tilde{f}(w) \stackrel{\text{def}}{=} \int_0^\infty F(\alpha) d\alpha \tag{3}$$

where the function F is defined as before.

Lovász extension

- For a given $w \in [0,1]^m$, it is easy to see that we can also define the Lovász extension as
$$\tilde{f}(w) = \mathbb{E}[f(e \in E : w(e_i) > \alpha)] \tag{4}$$
where α is uniform random variable in $[0,1]$.

(See Jan Vondrak, lecture notes, lecture 17, class notes for more extensions.)

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where α is uniform random variable in $[0,1]$.
- The convexity of the Lovász extension, the ease of minimizing convex functions, and the fact that we can recover f from $\tilde{f}$ via $f(A) = \tilde{f}(\mathbf{1}_A)$ corresponds to why SFM is possible in polynomial time (which was first shown by Grötschel, Lovász, and Schrijver in 1988 as part of their Ellipsoid method.

Choquet integral and aggregation

- Given the following form of aggregation

$$\mathrm{AG}(x) \stackrel{\text{def}}{=} \sum_{A:\mathbf{1}_A \in \mathcal{V}(x)} \left(\alpha_0^x(A) + \sum_{i=1}^{m} \alpha_i^x(A) x_i \right) \mathrm{AG}(\mathbf{1}_A) \tag{5}$$

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- We can define a canonical triangulation of the hypercube in terms of permutations of the coordinates. I.e., given some permutation σ, define

$$\text{conv}(\mathcal{V}_\sigma) = \big\{ x \in [0,1]^n | x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(m)} \big\} \tag{6}$$

Then these $m!$ blocks of the partition are called the canonical partitions of the hypercube. In this case, we have:

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Proposition 2.4

The above linear interpolation using the canonical partition yields the Lovász extension.

Polymatroid extreme points

Theorem 2.5

For a given ordering $E = (e_1, \dots, e_m)$ of E and a given E_i and x generated by E_i using the greedy procedure, then x is an extreme point of P_f

Polymatroid extreme points

- The greedy algorithm does more than solve $\max(wx : x \in P_f)$. We can use it to generate vertices of polymatroidal polytopes.

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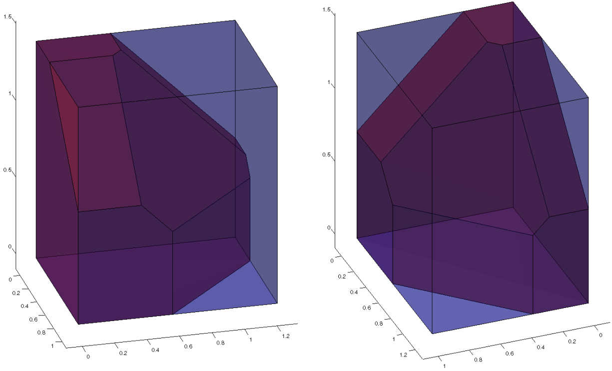

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$$B_f = P_f \cap \left\{ x \in \mathbb{R}_+^E : x(E) = f(E) \right\} \tag{7}$$

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$$B_f = P_f \cap \left\{ x \in \mathbb{R}_+^E : x(E) = f(E) \right\} \tag{7}$$
- Also, intuitively, we can continue advancing along the skeletal edges of the polytope to reach any other vertex, given the appropriate ordering. If we advance in all dimensions, we'll reach a vertex in B_f, and if we advance only in some dimensions, we'll reach a vertex in $P_f \setminus B_f$.

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- We formalize this next:

Polymatroid extreme points

- Given any arbitrary order of $E = (e_1, e_2, \ldots, e_m)$, define $E_i = (e_1, e_2, \ldots, e_i)$.

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- A vector x is generated by E_i using the greedy procedure as follows

$$x(e_1) = f(E_1) \tag{8}$$

$$x(e_j) = f(E_j) - f(E_{j-1}) \text{ for } 2 \leq j \leq i \tag{9}$$

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- An extreme point of P_f is a point that is not a convex combination of two other distinct points in P_f. Equivalently, an extreme point corresponds to setting certain inequalities in the specification of P_f to be equalities, so that there is a unique single point solution.

Polymatroid extreme points

Theorem 3.1

For a given ordering $E = (e_1, \ldots, e_m)$ of E and a given E_i and x generated by E_i using the greedy procedure, then x is an extreme point of P_f

Polymatroid extreme points

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Proof.

- We already saw that $x \in P_f$ (in Lecture 11, proof of Theorem 4.2).

□

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Proof.

- We already saw that $x \in P_f$ (in Lecture 11, proof of Theorem 4.2).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations

$$x(E_j) = f(E_j) \text{ for } 1 \le j \le i \quad (11)$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \quad (12)$$

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- And so on ...
- Also, since $x \in P_f$, for each i, we see that,

$$x(E_i) = f(E_i) \tag{13}$$

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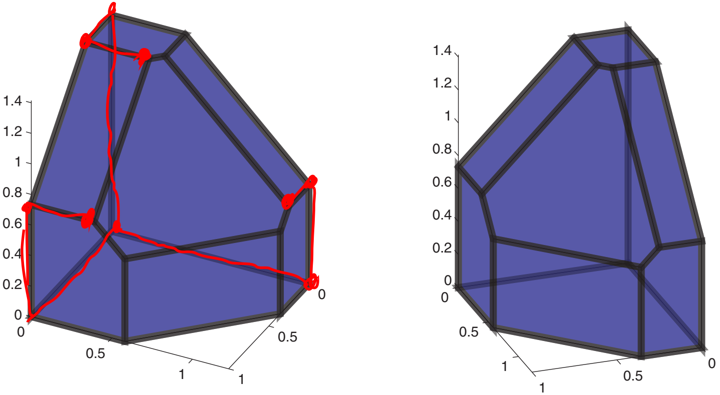
- Thus, the greedy procedure provides a modular function lower bound on f that is tight on all points E_i in the order.

Polymatroid extreme points

some examples



Polymatroid extreme points

- Moreover, we have

Corollary 3.2

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A))$, then x is generated using greedy by some ordering of A.

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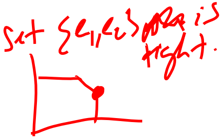
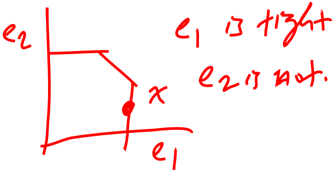
- Note, $\text{cl}(x) = \cup(A : x(A) = f(A))$ is the closure of x (recall that sets A such that $x(A) = f(A)$ are called tight, and such sets are closed under union and intersection, see Lecture 7, in proof of Theorem 4.3, starting Eq. 50).





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- Thus, $\text{cl}(x)$ is a tight set.

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- Note, $\text{cl}(x) = \cup(A : x(A) = f(A))$ is the closure of x (recall that sets A such that $x(A) = f(A)$ are called tight, and such sets are closed under union and intersection, see Lecture 7, in proof of Theorem 4.3, starting Eq. 50).
- Thus, $\text{cl}(x)$ is a tight set.
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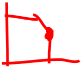
- For arbitrary x, $\text{supp}(x)$ is not tight, but for an extreme point, $\text{supp}(x)$ is.



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- Recall closure from Lecture 3: Given $A \subseteq E$, the closure or span of A, is defined by $\text{span}(A) = \sigma(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$ where r is matroid rank.

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- That is, we saw in Lecture 7 that for any $A, B \in \mathcal{D}(x)$, we have that $A \cup B \in \mathcal{D}(x)$ and $A \cap B \in \mathcal{D}(x)$.

Polymatroid Closure/Sat

- Now given:

$$\mathcal{D}(x) = \{A : A \subseteq E, x(A) = f(A)\} \quad (16)$$

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$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{18}$$

we must have $f(A) = f(B) = f(A \cup B) = f(A \cap B)$.

Polymatroid Closure/Sat

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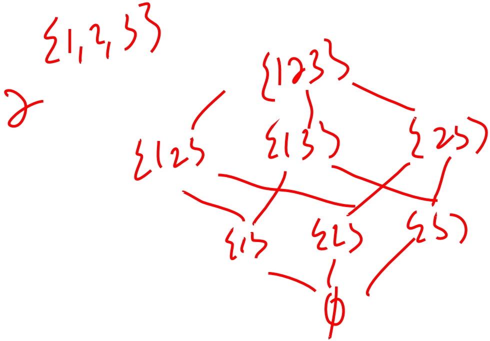

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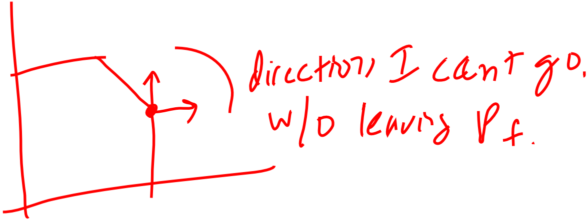

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- This generalizes matroid closure in the following way. Consider a matroid (E, r) and some $B \subseteq E$ and $\mathbf{1}_B$ with $\mathbf{1}_B$ a vertex in P_r. Then,

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But $\mathbf{1}_B(A) = \mathbf{1}_E(A \cap B)$ and so this can be never greater than either A or B, so any item $e \in A \setminus B$ that doesn't increase the rank of B will never get counted in $\mathbf{1}_E(A \cap B)$, nor will it be counted in $r(A)$.

$\mathbf{1}_E(A \cap B) = |A \cap B|$

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- Therefore, for matroid (E, r), we have $\sigma(B) = \text{sat}(\mathbf{1}_B)$.

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- Now, consider a matroid $(E, \mathcal{I}) = (E, r)$ and some $I \in \mathcal{I}$, then we have $\mathbf{1}_I \in P_r$ and

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- Thus, for a matroid, $\text{sat}(\mathbf{1}_I)$ is exactly the closure (or span) of I in the matroid.
- Also note, for $x \in P_f$ and polymatroidal f, $\text{sat}(x)$ is the maximal (by inclusion) minimizer of $f(A) - x(A)$, and thus in a matroid, $\text{span}(I)$ is the maximal minimizer of $r(A) - \mathbf{1}_I(A)$.

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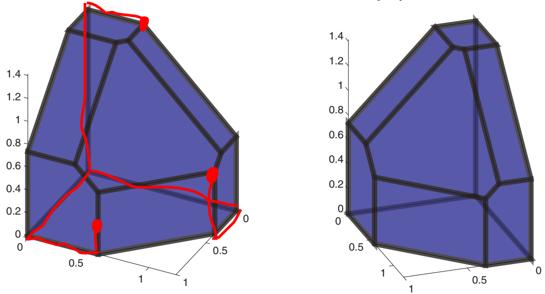

Dependence Function

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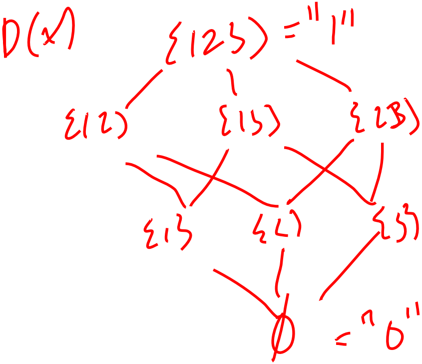

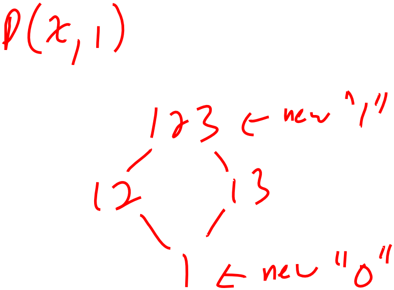

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- I.e., $\text{dep}(x,E)$ is the minimal element in $\mathcal{D}(x)$ that contains e (the minimal x-tight set containing e).

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(keep adding $r(A \setminus (I \cap A)) = r(A)$

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- then $\text{dep}(\mathbf{1}_I, e)$ must be a circuit since if it included more than a circuit, it would not be minimal in this sense.

Dependence Function

- Therefore, $\text{dep}(\mathbf{1}_I, e) = C(I, e)$ where $C(I, e)$ is the unique circuit contained in $I + e$ in a matroid, called the fundamental circuit of e in the independent set I.

by definition

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- We are thus free to take subsets of I as A, all of which contain e.
- Now, note in general $\text{dep}(x, e)$ for $x \in P_f$ and $e \in \text{sat}(x)$ is tight, by definition.

Dependence Function

- Summarizing,
  - For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated (x-tight) set w.r.t. x. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$
  - For $e \in \text{sat}(x)$, $\text{dep}(x, e)$ (fundamental circuit) is the minimal (common) saturated (x-tight) set w.r.t. x containing e.

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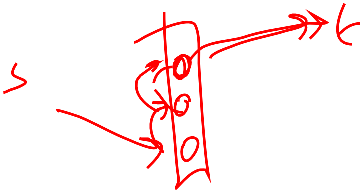
  - For $e \in \text{sat}(x)$, $\text{dep}(x, e)$ (fundamental circuit) is the minimal (common) saturated (x-tight) set w.r.t. x containing e.
- Recall, we have $C(I, e) \setminus e' \in \mathcal{I}$ for $e' \in C(I, e) \setminus \{e\}$. I.e., $C(I, e)$ consists of the elements that when removed recover independence.



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  - For $e \in \text{sat}(x)$, dep(x, e) (fundamental circuit) is the minimal (common) saturated (x-tight) set w.r.t. x containing e.
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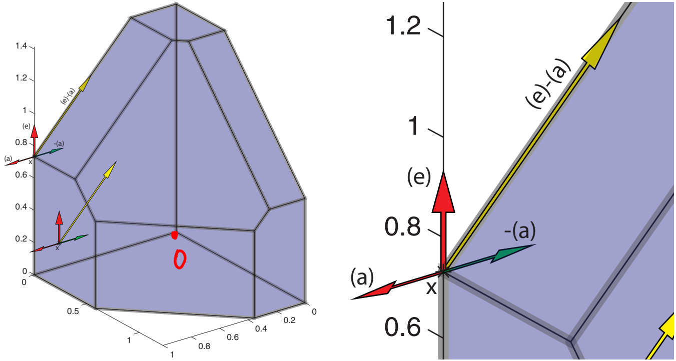
- Also, for $e \in \text{span}(I)$, we have that $I + e \notin \mathcal{I}$.
- Analogously, for $e \in \text{sat}(x)$, any $x + \alpha \mathbf{1}_e \notin P_f$ for $\alpha > 0$.
- But, analogous to the circuit case, is there an exchange property for dep(x, e)? Could move in this direction if we simultaneously move in another direction?

Dependence Function

- We can move neither in the (e) nor the (a) direction, but we can move in the (e) direction if we simultaneously move in the -(a) direction.



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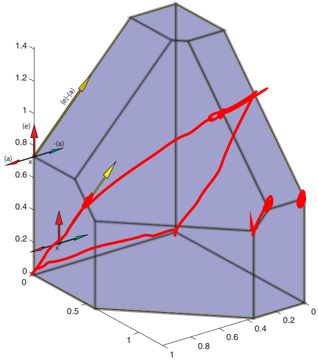

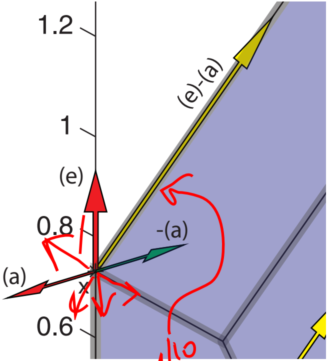

- This suggests

$$\text{dep}(x, e) = \{a : a \in E, \exists \alpha > 0 : x + \alpha(\mathbf{1}_e - \mathbf{1}_a) \in P_f\} \tag{32}$$

Extreme points by greedy

$\frac{1}{2}(f(A)+f(A)-f(v))^2$

Recall earlier theorem

Corollary 3.3

If x is an extreme point of P_f and $B \subseteq E$ is given such that $supp(x) \subseteq B \subseteq \mathrm{sat}(x)$, then x is generated using greedy by some ordering of A.

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- We know greedy finds an $v \in P_f$ such that for any c, $cv = \max(cx : x \in P_f)$, using aforementioned procedure.

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Proof. (sketch).

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- We also saw that greedy finds an extreme point.
- Choose $c = \mathbf{1}_{\text{supp}(x)}$.
- Then the greedy solution satisfies the same system of inequalities/equalities that x satisfies.

□

Outline

- We're next going to study lattices and submodular functions.
- In doing so, we'll better be able to understand certain properties of polymatroidal extreme points and ultimately SFM.

Partially ordered set

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- In a poset, for any $x, y, z \in V$ the following conditions hold (by definition):

For all $x, x \preceq x$.	(Reflexive)	(P1.)
If $x \preceq y$ and $y \preceq x,$ then $x = y$	(Antisymmetriy)	(P2.)
If $x \preceq y$ and $y \preceq z,$ then $x \preceq z.$	(Transitivity)	(P3.)

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- We can use the above to get other operators as well such as "less than" via $x \preceq y$ and $x \neq y$ implies $x \prec y$. Also, we get $x \succ y$ if not $x \preceq y$, etc. etc.

Partially ordered set

- There exists only one (unique minimal) element x which satisfies $x \preceq y$ for all y. Since if $x \preceq y$ for all y and $z \preceq y$ for all y then $z \preceq x$ and $x \preceq z$ implying $x = z$. We can name this element 0 (zero). The dual maximal element is called 1.

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- Given two elements, we need not have either $x \preceq y$ or $y \preceq x$ be true, i.e., these elements might not be comparable. If for all $x, y \in V$ we have $x \preceq y$ or $y \preceq x$ then the poset is **totally ordered**.

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- We define a set of elements $x_1, x_2, \ldots, x_n$ as a *chain* if $x_1 \preceq x_2 \preceq \cdots \preceq x_n$, which means $x_1 \preceq x_2$ and $x_2 \preceq x_3$ and $\ldots x_{n-1} \preceq x_n$. While we normally thing of the elements of a chain as distinct they need not be.

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Example 4.1

Let $V = \mathbb{Z}^+$ be the set of positive integers and let $x \preceq y$ mean that x is less than y in the usual sense. Then we have a poset that is actually totally ordered.

Partially ordered set

Example 4.2

Let V consist of all real single-valued functions $f(x)$ defined on the closed interval $[-1, 1]$, and let $g \leq f$ mean that $g(x) \leq f(x)$ for all $x \in [-1, 1]$. Again poset, but not total order.

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- Any subset of a chain is a chain.
- Two posets V_1 and V_2 are isomorphic if there is an isomorphism between them (i.e., a 1-1 order preserving (isotone) function that has an order preserving inverse). We write that two posets U and V are isomorphic by $U \simeq V$.

Partially ordered set

- **duality**. The dual poset is formed by exchanging $\leq$ with $\geq$. This is called the converse of a partial ordering. The converse of a PO is also a PO. We write the dual of V as V^d. U and V are dually isomorphic if $U = V^d$ or equivalently $V = U^d$. When $U = U^d$ then U is self-dual.

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The set $U = 2^E$ for some set E is a poset ordered by set inclusion. See Figure (C). Note that this U is self-dual.

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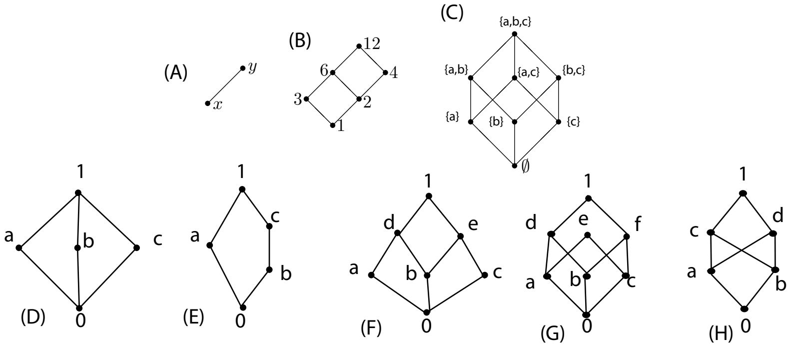
Example 4.4

Given an n-dimensional linear (Euclidean) space $\mathbb{R}^n$. A subset of $M \subseteq \mathbb{R}^n$ is an affine set if $(1-\lambda)x + \lambda y \in M$ whenever $x, y \in M$ and $\lambda \in \mathbb{R}$. A *linear subspace* of $\mathbb{R}^n$ is an affine set that contains the origin. Subspaces can be obtained via some A, b such that for every $y \in M$, $y = Ax + b$ for some $x \in \mathbb{R}^n$.
The set of all linear subspaces of $\mathbb{R}^n$ is a poset (ordered by inclusion), and such a set is self-dual.

Partially ordered set



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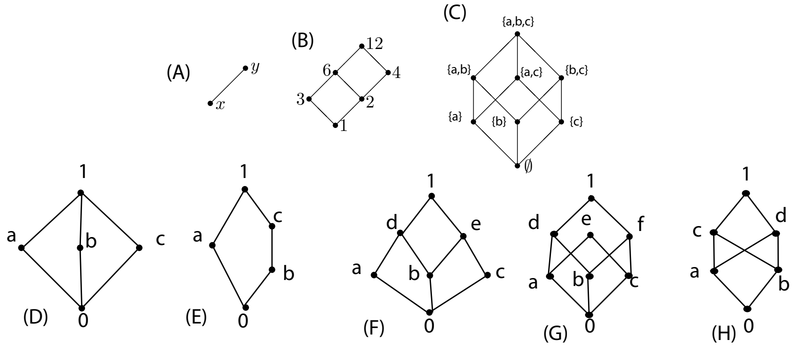

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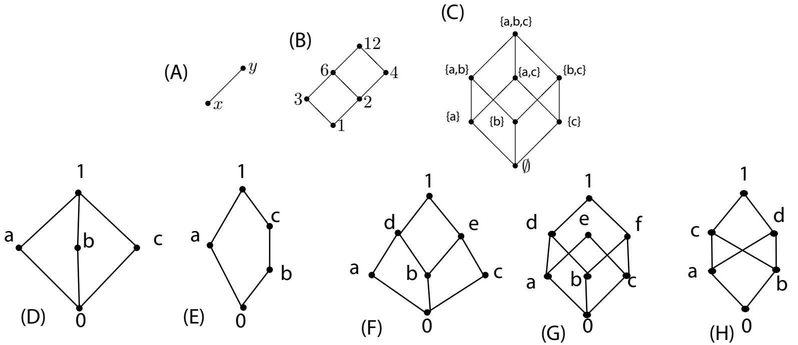

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- Hasse-diagram: We can draw a poset using a graph where each $x \in V$ is a node, and if $x \sqsubset y$ we draw y directly above x with a connecting edge, but no other edges.

Scratch Paper

Scratch Paper

Scratch Paper

Sources for Today's Lecture

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