EE595A – Submodular functions, their optimization and applications – Spring 2011

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Department of Electrical Engineering
Spring Quarter, 2011
http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 15 - May 19th, 2011

Announcements

- Homework 2 is due tonight at 11:45pm. All things in lectures marked "exercise"
- Again, all submissions must be done electronically, via our drop box. See the link `https://catalyst.uw.edu/collectit/dropbox/bilmes/14888`, or look at the homework on the web page.
- Last lecture, all annotations apparently lost (unless you are a PDF expert). Please email me any typos you discover in lecture 14!!

Class Road Map

We need to find one makeup lecture this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L11 (5/6): On SFM, polymatroid member & greedy, Lovász ext.
- L12 (5/11): Lovász ext. + polymatroid props.
- L13 (5/13): More polymatroids, start lattices
- L14 (5/18): lattices/submodular
- L15 (5/20): towards SFM
- L16 (5/25):
- L17 (5/27):
- L18 (6/1):
- L19 (6/3):
- L20: (6/9): 3-7:30pm (EEB-303)?

Polymatroids

Theorem 2.1

For a given ordering $E = (e_1, \ldots, e_m)$ *of* E *and a given* E_i *and* x *generated by* E_i *using the greedy procedure, then* x *is an extreme point of* P_f

Theorem 2.2

If x *is an extreme point of* P_f *and* $B \subseteq E$ *is given such that* $supp(x) \subseteq B \subseteq \text{sat}(x)$, *then* x *is generated using greedy by some ordering of* B.

Partially ordered set

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- In a poset, for any $x, y, z \in V$ the following conditions hold (by definition):

For all $x, x \preceq x$.	(Reflexive)	(P1.)
If $x \preceq y$ and $y \preceq x$, then $x = y$	(Antisymmetriy)	(P2.)
If $x \preceq y$ and $y \preceq z$, then $x \preceq z$.	(Transitivity)	(P3.)

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$$\text{If } x \preceq y \text{ and } y \preceq z, \text{ then } x \preceq z. \qquad \text{(Transitivity)} \qquad \text{(P3.)}$$

- The order $n(P)$ of a poset P is meant the (cardinal) number of its elements.

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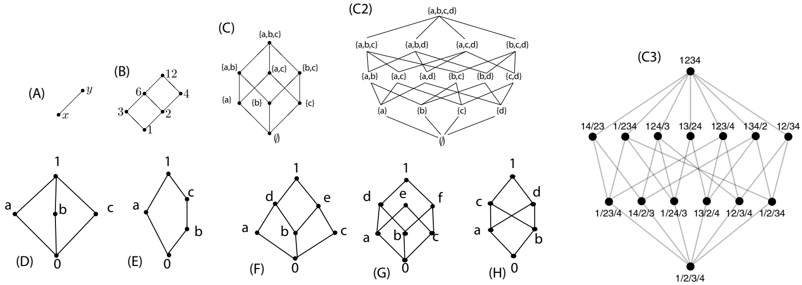

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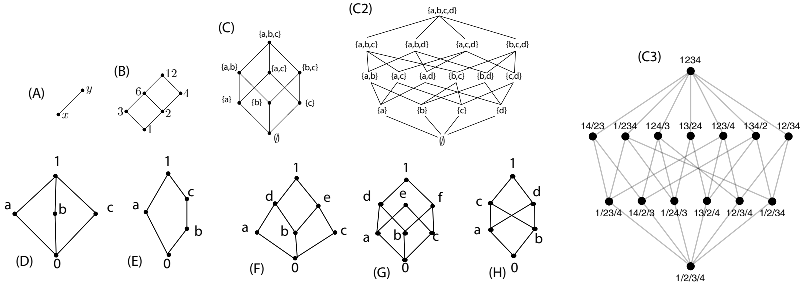

- Hasse-diagram: We can draw a poset using a graph where each $x \in V$ is a node, and if $x \sqsubset y$ we draw y directly above x with a connecting edge, but no other edges.

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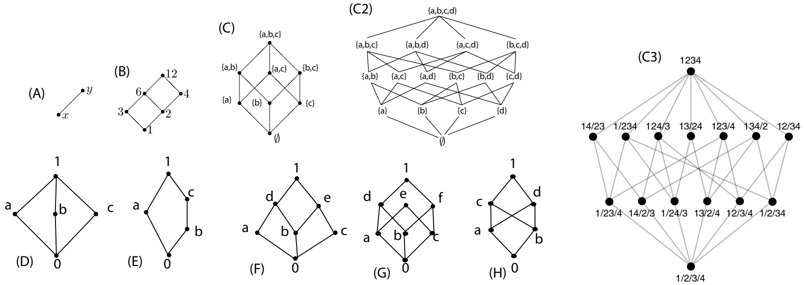

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Theorem 2.3

Every non-empty finite subset $X \subseteq V$ has a minimal (and maximal) element.

Partially ordered set

Theorem 2.4

Every non-empty finite subset $X \subseteq V$ has a minimal (and maximal) element.

Proof.

Let $X = \{x1, \ldots, x_n\}$. Define $m_1 = x_1$ and

$$m_k = \begin{cases} x_k & \text{if } x_k \prec m_{k-1} \\ m_{k-1} & \text{otherwise} \end{cases} \quad (1)$$

Then we have constructed $m_n \preceq m_{n-1} \preceq \cdots \preceq m_1$ meaning there is no m_k for $k < n$ such that $m_k \prec m_n$. Let $M = \{m_1, \ldots, m_n\}$. By construction, we also have that there is no $x \in X$ with $x \prec m_n$, thus m_n is minimal. □

Partially ordered set

- Given a poset V, the length $\ell(V)$ is defined to be the l.u.b. of the lengths of any chains in V. That is, $\ell(V)$ is the least upper bound, i.e., smallest number not less than any chain length in V.

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- The height or *dimension* of an element $x \in V$, or $l = h(x)$ is the l.u.b. of the lengths of the chains $0 = x_0 \prec x_1 \prec \dots x_l = x$ between 0 and x. Note that $h(1) = \ell(V)$ when they exist. $h(x) = 1$ iff $0 \sqsubset 1$ and such elements (with unit height) are called "atoms" or "points" or "(ground) elements".

Partially ordered set

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(or JDCC) All maximal length chains between the same endpoints have the same finite length.

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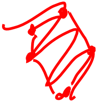
- With JDCC, if $x \sqsubset y$ then $h(x) + 1 = h(y)$.



Partially ordered set

- With JDCC, element x has height or *rank* $h(x)$. The height (rank) function in this case is unique. If $x \preceq y$ then $\ell(x, y) = h(y) - h(x)$ is the length between x and y.

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- We say a poset is "graded" if it is graded by the height function.

Lattice defined

- Given $X \subset V$, $y \in V$ is an upper bound of X if $x \preceq y$ for all $x \in X$. Note that y need not be in X. If y is a least upper bound (l.u.b. X or just sup$X$), then $y \preceq z$ for any other upper bound z. The l.u.b. if it exists is unique since if y_1 and y_2 are both l.u.b.'s then $y_1 \preceq y_2$ and $y_2 \preceq y_1$, or $y_1 = y_2$. Dual definitions for lower bound and greatest lower bound (g.l.b.).

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Definition 2.7 (lattice)

A *lattice* is a poset V such that any two elements $x, y \in v$ have a g.l.b. or meet denoted by $x \wedge y \in V$, and also have a l.u.b. or join denoted by $x \vee y \in V$. A lattice is complete when all subsets $X \subseteq V$ have both a l.u.b. and a g.l.b. (note that join and meet is defined on pairs, but l.u.b. and g.l.b. can be defined on any subset of V, even of size 1).

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- Note again, that such l.u.b.'s and g.l.b.'s are unique if they exist.

Lattices

- Any finite lattice or lattice of finite length is complete. Note that the reverse need not hold (a complete lattice need not be finite). The reals are not complete but the extended reals are complete. The rationals are not complete (but the rationals extended with a $\pm\infty$ is complete). 2^E for some set E is complete. Note that E can be countably or uncountably infinite.

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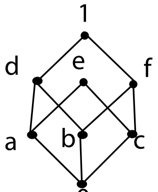
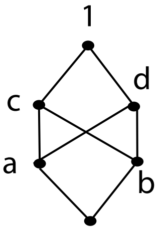
- All of the figures above are lattices except for (G) and (H). (G) is not a lattice since for example $e \vee f$ does not exist, nor does any join with e exist. (H) is not a lattice since there are two joins for a and b.



(G)



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- The dual of a lattice is a lattice, and the dual of a complete lattice is a complete lattice.
- In a chain, $x \wedge y$ is the smaller of the two, and $x \vee y$ is the larger of the two.

Lattices

Definition 2.8 (sublattice)

A sublattice of a lattice is a subset $X \subseteq V$ such that join and meet are closed within X (for all $x, y \in X$, $x \vee y \in X$ and $x \wedge y \in X$). A sublattice is a lattice.

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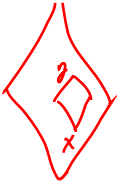
- Given any $x \preceq y$, then all elements $\{z : x \preceq z \preceq y\}$ form a sublattice. We note that in such case, we say that $[x, y]$ form a (closed) interval in the lattice, and we have that the (closed) interval $[x, y]$ of all elements $z \in L$ such that $x \preceq z \preceq y$ is a sublattice.



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- Obviously, 2^E for some set E is a lattice, with join/meet being union/intersection. See Figure(C).

Lattices

Theorem 2.9

In any poset V, the operations of meet and join satisfy the following laws, whenever the associated expressions exist.

$$x \wedge x = x, x \vee x = x \quad \text{(Idempotent)} \quad \text{(L1)}$$

$$x \wedge y = y \wedge x, x \vee y = y \vee x \quad \text{(Commutative)} \quad \text{(L2)}$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z, x \vee (y \vee z) = (x \vee y) \vee z \quad \text{(Associative)} \quad \text{(L3)}$$

$$x \wedge (x \vee y) = x \vee (x \wedge y) = x \quad \text{(Absorption)} \quad \text{(L4)}$$

$$x \preceq y \iff x \wedge y = x \text{ and } x \vee y = y \quad \text{(Consistency)} \quad \text{(CON)}$$

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Note the above works for posets, not necessary for it to be a lattice.

Lattices

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Theorem 2.11

In any lattice, the operations of join and meet are order-preserving in the following sense:

$$y \preceq z \Rightarrow x \wedge y \preceq x \wedge z \text{ and } x \vee y \preceq x \vee z \tag{3}$$

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Theorem 2.12

In any lattice, the following distributive inequalities hold for all $x, y, z \in V$:

$$x \wedge (y \vee z) \succeq (x \wedge y) \vee (x \wedge z) \tag{4a}$$

$$x \vee (y \wedge z) \preceq (x \vee y) \wedge (x \vee z) \tag{4b}$$

Distributive Inequalities

Theorem 2.13

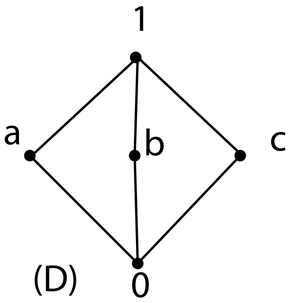
In any lattice, the following distributive inequalities *hold for all* $x, y, z \in V$:

$$x \wedge (y \vee z) \succeq (x \wedge y) \vee (x \wedge z) \tag{5a}$$

$$x \vee (y \wedge z) \preceq (x \vee y) \wedge (x \vee z) \tag{5b}$$

Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).



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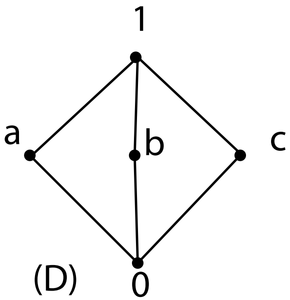

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- For example, in (D), we have that $a \wedge (b \vee c) = a \wedge 1 = a$ but $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$ and obviously $a \succ 0$.

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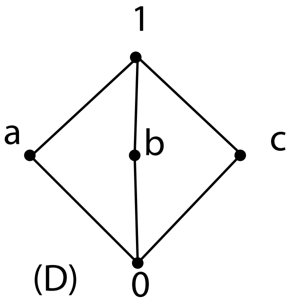

- For example, in (D), we have that $a \wedge (b \vee c) = a \wedge 1 = a$ but $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$ and obviously $a \succ 0$.
- Also, in (D) we have $a \vee (b \wedge c) = a \vee 0 = a \prec (a \vee b) \wedge (a \vee c) = 1 \wedge 1 = 1$.

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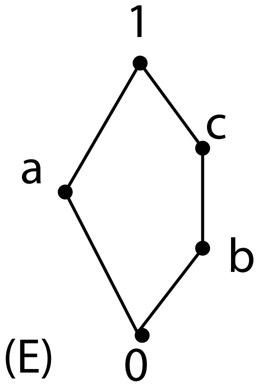

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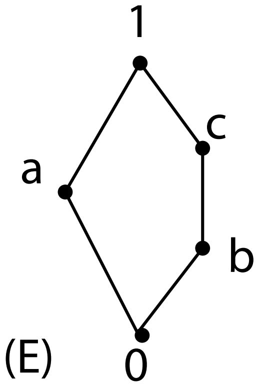

- In (E), we have that $c \wedge (a \vee b) = c \wedge 1 = c \succ (c \wedge a) \vee (c \wedge b) = 0 \vee b = b$

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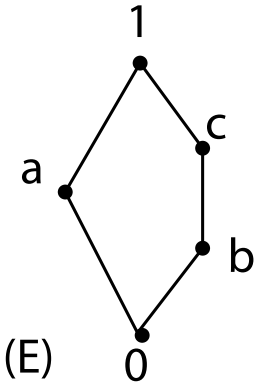

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- Also, in (E), we have $b \vee (a \wedge c) b \vee 0 = b \prec (b \vee a) \wedge (b \vee c) = 1 \wedge c = c$

Modular inequality

Theorem 2.14

In any lattice, the following modular inequalities holds for all $x, y, z \in V$*:*

$$x \preceq z \Rightarrow x \vee (y \wedge z) \preceq (x \vee y) \wedge z \tag{6}$$

Distributive Lattices

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Theorem 3.1

In any lattice, the following are equivalent:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \tag{7a}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall x, y, z \tag{7b}$$

✓ $a \cdot (b+c) = ab + ac$

✗ $a + b \cdot c \neq (a+b) \cdot (a+c)$

$\therefore$ $\vee \equiv +$, $\wedge \equiv \cdot$ $\}$ $\Rightarrow$ not a distr lattice

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It is important to note the $\forall x, y, z$ since this is not true only for individual elements. Note moreover that this means that the operators $\vee = +$ and $\wedge = \cdot$ do not form a lattice over $\mathbb{R}$.

Distributive Lattices

Theorem 3.2

In any lattice, the following are equivalent:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \tag{8a}$$

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Distributive Lattices

Theorem 3.2

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Proof.

Take as given the 2nd equation and show the first. Then

$$\begin{aligned}
(x \wedge y) \vee (x \wedge z) &= [(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z] && \text{by the 2nd eq} && (9)\\
&= x \wedge [(x \wedge y) \vee z] && x \wedge y \preceq x && (10)\\
&= x \wedge [(x \vee z) \wedge (y \vee z)] && \text{by the 2nd eq} && (11)\\
&= x \wedge (x \vee z) \wedge (y \vee z) && \text{associative} && (12)\\
&= x \wedge (y \vee z) && x \vee z \succeq x && (13)
\end{aligned}$$

□

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Example 3.3

Let $V = \mathbb{Z}^+$ be the set of positive integers and let $x \preceq y$ mean that x divides y. I.e., $2 \preceq 4$ but $2 \not\preceq 5$. Then this is lattice with $x \vee y = \text{l.c.m.}(x, y)$ and $x \wedge y = \text{g.c.d.}(x, y)$. It is also distributive. Again consider figure (B).

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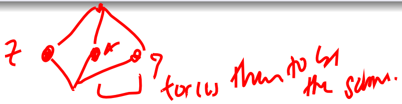
Theorem 3.4 (identity)

In a distributive lattice, if $z \wedge x = z \wedge y$ and $z \vee x = z \vee y$ then $x = y$.



Modular Lattices

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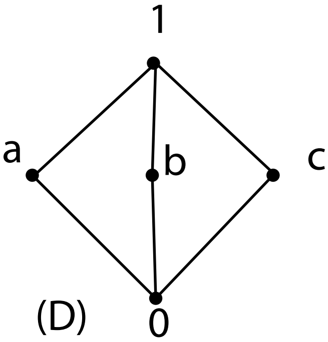
- Easy way to remember. x,y,z and $x \vee (y \wedge z) = (x \vee y) \wedge z$
- The term "modular" comes from abstract algebra, where a R-module is an abstract system that generalizes $(\mathbb{R}, \mathbb{R}^n)$ (i.e., a vector field with scalar multiplication). An R-module ends up being a lattice that satisfies this identity.

Modular Lattices

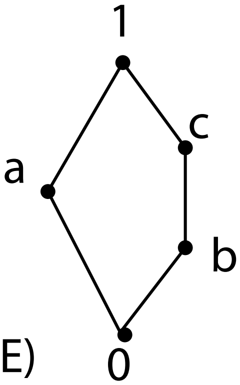
- Not every lattice is modular. Figure (D) is modular but not distributive. We already saw that (D) is not distributive since it is strict for certain assignments. It is modular though.



(D)

Modular Lattices

- Figure (E) is neither modular nor distributive. We saw that it was not distributive since it achieved strictness in the distributive inequalities. It is not modular since: take $b \preceq c$, then $b \vee (a \wedge c) = b \vee 0 = b \prec (b \vee a) \wedge c = 1 \wedge c = c$, so modular equality is violated.



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Any non-modular lattice V contains the lattice in Figure (E) as a sublattice.

- Thus, the structure (E) is fundamental to non-modular lattices.

Modular Lattices

Theorem 4.3

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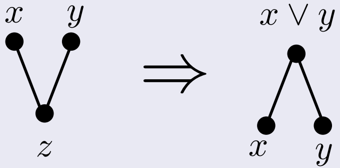
Upper-Semimodularity if x and y cover z and $x \neq y$ then $x \vee y$ covers both x and y, and



Thus, upper-semimodularity means that if $z \sqsubset x$ and $z \sqsubset y$, and if $x \neq y$, then $x \sqsubset (x \vee y)$ and $y \sqsubset (x \vee y)$.

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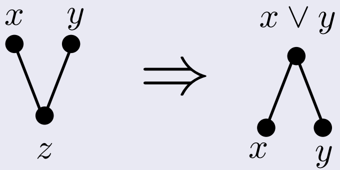
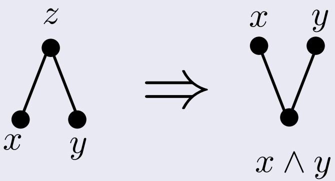


Lower-Semimodularity if z covers x and y and $x \neq y$ then x and y both covers $x \wedge y$.



Thus, lower-semimodularity means that if $x \sqsubset z$ and $y \sqsubset z$, and if $x \neq y$, then $(x \wedge y) \sqsubset x$ and $(x \wedge y) \sqsubset y$.

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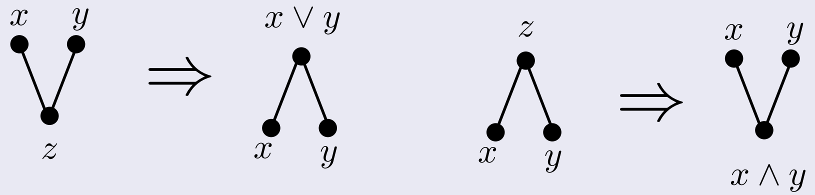

As we will see, the first equation implies submodularity on the dimension (height function) and the second equation implies supermodularity on the dimension (height) function. Both together imply modularity on the dimension function.

Semi-modular/Submodular Lattices

Theorem 5.1

Let L be a finite lattice. The following two conditions are equivalent:

(i) *L is graded, and the height function $h(\cdot)$ of L satisfies the (what we know as the submodular) inequality for all $x, y \in L$.*

$$h(x) + h(y) \geq h(x \vee y) + h(x \wedge y) \tag{14}$$

(ii) *If x and y both cover z, then $x \vee y$ covers both x and y*

≡ upper semimodular.

Semi-modular/Submodular Lattices: (i) ⇒(ii)

vee implies cup

$$h \text{ submodular} \Rightarrow \left\{ (z \sqsubset x, z \sqsubset y) \Rightarrow (x \sqsubset (x \vee y)), (y \sqsubset (x \vee y)) \right\}.$$

- Suppose x and y cover z.

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Semi-modular/Submodular Lattices: (i) ⇒(ii)

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- Note that if x and y cover z then since L is a lattice, $z = x \wedge y$.

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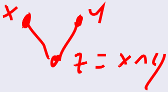
- Note that if x and y cover z then since L is a lattice, $z = x \wedge y$.
- Then we have $h(x) = h(y) = h(x \wedge y) + 1$.



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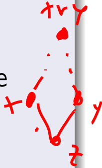
- Then we have $h(x) = h(y) = h(x \wedge y) + 1$.
- Also, since x and y are distinct, and since they both cover z we can't have (w.l.o.g.) $x \preceq y$, and thus $h(x \vee y) > h(x) = h(y)$.



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- Hence by (i), we have

$$h(x) + h(y) - h(x \wedge y) \geq h(x \vee y) > h(x \wedge y) + 1 \qquad (15)$$

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$$h(x \wedge y) + 2 \geq h(x \vee y) > h(x \wedge y) + 1 \tag{16}$$

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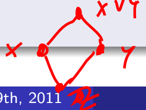
- giving $h(x \vee y) = h(x \wedge y) + 2 = h(x) + 1 = h(y) + 1$, so that $x \vee y$ covers both x and y.

. . .



Semi-modular/Submodular Lattices: (ii) $\Rightarrow$ (i)

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grading &

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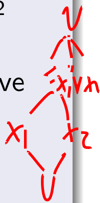

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- Hence L is graded (i.e., every maximal chain has the same length, i.e., JDCC holds).

. . .

Semi-modular/Submodular Lattices: (ii) $\Rightarrow$ (i)

$\Big\{(z \sqsubset x, z \sqsubset y) \Rightarrow (x \sqsubset (x \vee y)), (y \sqsubset (x \vee y))\Big\} \Rightarrow h$ submodular.

- Now suppose there is a pair $x, y \in L$ violating the submodularity inequality, i.e., with

$$h(x) + h(y) < h(x \vee y) + h(x \wedge y) \tag{17}$$

and choose such a pair first with $\ell(x \wedge y, x \vee y)$ minimal, and then (second) with $h(x) + h(y)$ minimal.

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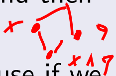

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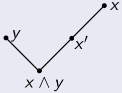
- Thus assume that $x \wedge y \prec x' \prec x$, say (w.l.o.g.) 

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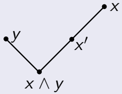
- Thus assume that $x \wedge y \prec x' \prec x$, say (w.l.o.g.) 

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- By the minimality of $\ell(x \wedge y, x \vee y)$ and $h(x) + h(y)$, we have

$$h(x') + h(y) \geq h(x' \wedge y) + h(x' \vee y). \tag{18}$$

...

Semi-modular/Submodular Lattices: (ii) $\Rightarrow$ (i)

$\Big\{ (z \sqsubset x, z \sqsubset y) \Rightarrow (x \sqsubset (x \vee y)), (y \sqsubset (x \vee y)) \Big\} \Rightarrow h$ submodular.

- Now $x' \wedge y = x \wedge y$, so Eq. 17 and Eq. 18 together imply that

$$h(x) + h(x' \vee y) < h(x') + h(x \vee y). \tag{19}$$

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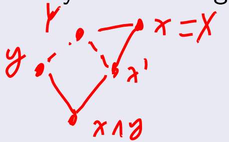

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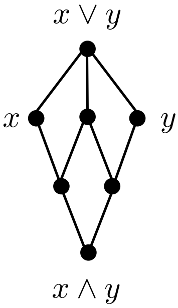
- The proof is complete.

Submodular Lattices

The next figure is an example of an upper-semimodular (or a "submodular") lattice over 7 elements.



height
3
2
1
0

submodularity
$h(x)+h(y)$
$> h(x \vee y)$
$+ h(x \wedge y)$

$2+2>3+0$

- This lattice is not modular since $x \vee y$ covers x and y, but x and y don't cover $x \wedge y$.

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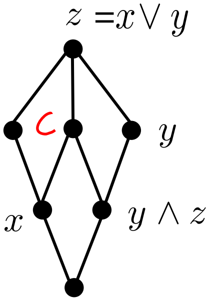

- Also notes, this violates the modular equality $(\forall x, y, z,\ \ x \preceq z \Rightarrow (x \vee (y \wedge z) = (x \vee y) \wedge z))$.

$= x \vee (y \wedge z) = c$

$(x \vee y) = z$

$z \wedge z = z$

$c \neq z$

Submodular Lattices

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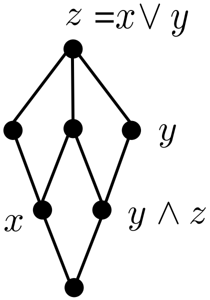
$h(x)+h(y) \geq h(x \vee y)$



- Also notes, this violates the modular equality $(\forall x, y, z, \;\; x \preceq z \Rightarrow (x \vee (y \wedge z) = (x \vee y) \wedge z))$.
- Flip it up side down to get a lower-semimodular (or "supermodular") lattice.

Ideal in a Lattice

Definition 6.1 (ideal)

An ideal is a nonvoid subset J of a lattice L with the properties

$$\forall a \in J, x \in L, \quad x \preceq a \Rightarrow x \in J \tag{20}$$

$$\forall a \in J, \quad b \in J \Rightarrow a \vee b \in J. \tag{21}$$

The dual concept (in a lattice) is called a dual ideal (or a meet ideal).

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Example 6.3

In 2^E, take any $A \subseteq E$, then $L(A) = \{B : B \subseteq A\}$ is an ideal in a set lattice.

Ideal in a Lattice

Definition 6.4

Given an element $a \in L$ in a lattice, the set $L(a)$ of all elements $\{x : x \preceq a, x \in L\}$ is an ideal, and is called a principle ideal.

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The set of all ideals'of any lattice L, ordered by inclusion, itself forms a lattice. The set of all principal ideals in L forms a sublattice of this lattice, which is isomorphic with L.

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Example 6.6

Consider 2^E. Then for any $A \subseteq E$, we see that $L(A) = \{B : B \subseteq A\}$ is an ideal. Also, we can see that the set of sets $\{L(A) : A \subseteq E\}$ is isomorphic to 2^E and also forms a lattice.

Complement and Complemented Lattices

Definition 6.7

A lattice with a 0 and 1 is complemented if for all $x \in L$ there exists a $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$.

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Proposition 6.9

In a complemented modular lattice of finite length, every element is the join of those elements which it contains.

Boolean Lattices

Definition 6.10

A boolean lattice is a complemented distributive lattice.

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Theorem 6.11

In any boolean lattice, each element x has a unique complement x'. Moreover, we have

$$x \wedge x' = 0, \qquad x \vee x' = 1 \tag{L1}$$

$$(x')' = x, \tag{L2}$$

$$(x \wedge y)' = x' \vee y', \qquad (x \vee y)' = x' \wedge y' \tag{L3}$$

Join Irreducible

Definition 6.12

An element x of a lattice is called join irreducible if $y \vee z = x$ implies $y = x$ or $z = x$ (ie, if x is the join of two elements, it must be one of those elements).

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Proposition 6.14

In any complemented modular lattice, all join irreducible elements are atoms.

Ring of sets

Definition 6.15 (ring family).

A ring of sets is a family Φ of subsets of a set E which contains with any two sets S and T also their (set-theoretic) intersection $S \cup T$ and union $S \cap T$. A field of sets is a ring of sets which contains with any S also its set complement $E \setminus S$'

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- Thus, any ring of sets under the natural ordering $S \subset T$ forms a distributive lattice.

Join irreducible, ground elements, Boolean lattices

Theorem 6.16

Let L be any distributive lattice of length n. Then the poset X of join-irreducible elements $x \succ 0$ has order n and, moreover, $L \cong 2^X$

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Theorem 6.17

Every Boolean lattice of finite length n is isomorphic with the field of all subsets of a set of $|E| = n$ elements, namely 2^E.

supp,sat, and dep

- For $x \in P_f$, $\text{supp}(x) = \{e : x(e) \neq 0\}$

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- For $x \in P_f$, $\text{supp}(x) = \{e : x(e) \neq 0\}$
- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated (x-tight) set w.r.t. x. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$. That is,

$$\text{cl}(x) \stackrel{\text{def}}{=} \text{sat}(x) \stackrel{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x)\} \tag{22}$$

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$\mathcal{D}(x) \cap \{A : e \in A\}$

- For $e \in \text{sat}(x)$, $\text{dep}(x, e)$ (fundamental circuit) is the minimal (common) saturated (x-tight) set w.r.t. x containing e. That is,

$$\text{dep}(x,e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

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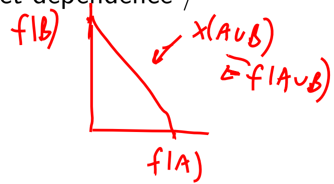
$f(A \cup B) = f(A) + f(B)$



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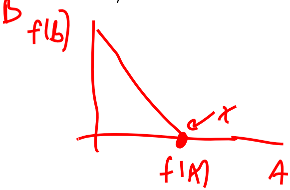

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- $\text{sat}(x) = \cup\{A : x(A) = f(A)\}$ and since $x(A \cup B) = x(A) = f(A) = f(A \cup B)$, here, $\text{sat}(x) = A \cup B$.

supp,sat, and dep

- Now, $\text{sat}(x)$ is tight, and corresponds to the largest member of the distributive lattice $\mathcal{D}(x) = \{A : x(A) = f(A)\}$.
- $\text{supp}(x)$ is not nec. tight, but for an extremal point, $\text{supp}(x)$ is tight (we see this from definition of extremal point defined by $x(E_i) = f(E_i)$.
- For $x \in P_f$, with x extremal, is $\text{supp}(x) = \text{sat}(x)$?
- Consider the case where disjoint $A, B \subseteq E$, we have $f(A) = f(B) = f(A \cup B)$ (meaning perfect dependence / redundancy).
- Suppose $x \in P_f$ has $x(A) > 0$ but $x(B) = 0$.
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- In general, $\text{sat}(x) \supseteq \text{supp}(x)$.

(for modular functions, always equal).

dep and partial order

- Consider $x \in P_f$, and consider the following set

$$\mathcal{D} = \{\text{dep}(x, e) : e \in \text{sat}(x)\} \tag{26}$$

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- Moreover, define a partial order on $\mathcal{D}$ as if $A, B \in \mathcal{D}$, then $A \preceq B$ iff $A \subseteq B$.
- We're going to use this partial order to define a partial order on all elements of sat(x).

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Theorem 7.1

If $x \in P_f$ is an extreme point, then $\preceq$ is a partial order on $\text{sat}(x)$ where for $a, e \in \text{sat}(x)$, the order $\preceq$ is defined by: $a \preceq e$ iff $a \in \text{dep}(x, e)$.

Scratch Paper

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Sources for Today's Lecture

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