EE595A – Submodular functions, their optimization and applications – Spring 2011

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Department of Electrical Engineering
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http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 17 - May 27th, 2011

Announcements

- Last lecture, and final presentations, will take place Thursday, June 9th, from 3-7:30pm. The lecture will be from 3:00-5:00pm, and the final presentations will be from 5:00-7:30pm. Please bring dinner.

Class Road Map

We need to find one makeup lecture this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L11 (5/6): On SFM, polymatroid member & greedy, Lovász ext.
- L12 (5/11): Lovász ext. + polymatroid props.
- L13 (5/13): More polymatroids, start lattices
- L14 (5/18): lattices/submodular
- L15 (5/20): lattices, → SFM.
- L16 (5/25): → SFM
- L17 (5/27):
- L18 (6/1):
- L19 (6/3):
- L20: (6/9): 3-7:30pm (EEB-303)?

dep and partial order

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- Moreover, define a partial order on $\text{DEP}(x)$ as follows: if $A, B \in \text{DEP}(x)$, then $A \preceq B$ iff $A \subseteq B$.

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- We're going to use this partial order to define a partial order on all elements of $\text{sat}(x)$.
- Now recall $\mathcal{D}(x) = \{A : x(A) = f(A)\}$ forms a distributive lattice. What is the natural partial order?

dep and partial order

- Now in any distributive lattice L, consider its join-irreducibles $\mathcal{J}$ (i.e., any element $A \in \mathcal{J}$ can't be represented as a join of any other two elements in L).
- We saw that if the lattice has length n, then $\mathcal{J}$ will have exactly n elements (in the Boolean case, these are atoms/ground elements), and each element in $\mathcal{J}$ is partially ordered by the lattice partial order.
- Moreover, we saw any element can be "generated" by joining the join-irreducible elements.

dep and partial order

- Now any element in $\text{DEP}(x)$ (for x extreme) can't be represented by the join of two other elements in $\text{DEP}(x)$, since the minimal tight sets containing e would not be generated by merging two minimal tight sets containing, say, a, and b, where all of a, b, e are unequal.
- Thus, considering $\mathcal{D}(x)$ as a distributed lattice, then $\text{DEP}(x)$ are the join-irreducibles.
- And the order $\preceq$ defined earlier is the natural order w.r.t. this lattice and its join-irreducibles.

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- Let $x \in P_f$ again be an extreme point, and let it be generated by an ordering of $B = (e_1, e_2, \dots, e_k) \subseteq E$ with $B_i = (b_i, b_2, \dots, b_i)$ a partial order w.r.t. ordered items B (B and $B_i, \forall i$ are ordered sets).

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- Recall, the equation for x is of the form $x(e) = 0$ for some e and $x(A) = f(A)$ for some A (see earlier). Specifically, we have that $x(E \setminus B) = 0$ and, for $i = 1 \dots k$, $x(B_i) = f(B_i)$.

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- We also have that $\text{supp}(x) \subseteq B$ due to monotonicity.
- Thus, for any $d, e \in \text{supp}(x) \subseteq B$, there is a tight set containing one but not the other. Specifically, let $d = e_i$ and $e = e_j$ with $j > i$. Then non-zero B_i (i.e., $B_i \cap \text{supp}(x)$) contains d but not e (note, vice versa is not true).

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- Thus, for any $d, e \in \text{supp}(x) \subseteq B$, we have $\text{dep}(x, d) \neq \text{dep}(x, e)$.
- Moreover, for any $e \in B$, we can have that $\text{dep}(x, e) = B_i$ where $e = e_i$. *This point is further clarified in the next slide.*

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- On the other hand, consider the extreme vector $x^{(i)} \in \mathbb{R}^E$ with

$$x^{(i)}(e) = \begin{cases} x(e) & \text{if } e \in B_i \\ 0 & \text{else} \end{cases} \tag{2}$$

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- On the other hand, $B_i \not\subset \text{dep}(x, e_i)$ due to $\text{dep}(x, e_i)$'s minimality.
- Therefore, we see that in general, $\text{dep}(x, e_i) \subseteq B_i$.

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- For $e \in B$ and $B' \in \mathcal{B}(x)$, let $1 \leq e(B') \leq |B'|$ be e's position in B'.
- Then $dep(x, e_i) = B_j^{e_i}$ where

$$B^{e_i} \in \operatorname*{argmin}_{B' \in \mathcal{B}(x)} e_i(B') \qquad \text{and also} \qquad \left|\operatorname*{argmin}_{B' \in \mathcal{B}(x)} e_i(B')\right| = 1 \qquad (3)$$

is ordered, and j is the position of e_i in B^{e_i}. Follows from iff relationship between extremal points and greedy algorithm, and since $\text{dep}(x, e_i)$ is the unique "0" element of a distributive lattice.

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  - Clearly, $\text{dep}(x, d) \subset \text{dep}(x, e) \Rightarrow d \in \text{dep}(x, e)$.

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  - Clearly, $\text{dep}(x, d) \subset \text{dep}(x, e) \Rightarrow d \in \text{dep}(x, e)$.
  - Also $d \in \text{dep}(x, e)$ means $\text{dep}(x, d) \subseteq B_k^e$ where $k = d(B^e)$ is the position of d in B^e (since B_k^e is a tight set containing d), but it must be that $k < j$ (since B_j^e is the smallest tight set containing e and the j'th position of B_j^e is e).

dep and partial order

- Also, for polymatroidal f, we saw earlier that for each $e \in \text{sat}(x) \setminus \text{supp}(x)$, the set $\text{supp}(x) + e$ is also tight. This follows since $x(\text{supp}(x) + e) = x(\text{supp}(x))$ but e is dependent on $\text{supp}(x)$ so that $f(\text{supp}(x) + e) = f(\text{supp}(x))$.

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- This gives further support to the phrase "dependence function", namely $\text{dep}(x, e) \setminus \{e\} = B_j$ is the smallest set that renders e dependent (again, like the fundamental circuit of a matroid).
- Thus, we have 1-1 mapping between all elements of $\text{sat}(x)$ and $\text{DEP}(x) = \{\text{dep}(x, e) : e \in sat(x)\}$.

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- Therefore, the partial order on $\text{DEP}(x)$ can be used to define a partial order on $\text{sat}(x)$.

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- We already saw, this happens iff $d \in \text{dep}(x, e)$.
- Thus, we can define a partial order on the elements of $\text{sat}(x)$ as follows:

Definition 2.1 (partial order on elements of $\text{sat}(x)$)

For $d, e \in \text{sat}(x)$, we have

$$d \preceq e \Leftrightarrow d \in \text{dep}(x, e) \tag{4}$$

dep and partial order

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Theorem 3.1

If $x \in P_f$ is an extreme point, then $\preceq$ is a partial order on $\text{sat}(x)$ where for $a, e \in \text{sat}(x)$, the order $\preceq$ is defined by: $a \preceq e$ iff $a \in \text{dep}(x, e)$.

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- In fact, we have a stronger result that extreme points are characterized by this construct:

Theorem 3.2

$x \in P_f$ is an extreme point, iff $\text{supp}(x) \subseteq \text{sat}(x)$ and $\text{dep}(x, a) \neq \text{dep}(x, b)$ for every pair of distinct points $a, b \in \text{sat}(x)$.

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- If f is strictly submodular, then the above order $\preceq$ is a total order on $\text{sat}(x)$.

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Definition 3.3

Given a partial order $\preceq$ and an ordered set $B = (e_1, e_2, \ldots, e_k)$, then B is compatible with $\preceq$ if for all $i < j$ we have that $e_i \preceq e_j$.

the partial order of extreme points

Theorem 3.4

Let x be an extreme point of P_f and $\preceq$ be its partial order. Let $B \subseteq E$ be an ordered set. Then B generates x using the greedy algorithm iff we have $\text{supp}(x) \subseteq B \subseteq \text{sat}(x)$ and B is compatible with $\preceq$.

Proof.

- Generate ⇒ Compatible: Let B generate x

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- Generate $\Rightarrow$ Compatible: Let B generate x
- Then $\text{supp}(x) \subseteq B$.
- Also, since B is tight, $B \in \mathcal{D}(x)$, so $B \subseteq \text{sat}(x)$.

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- Moreover, $B_j \in \mathcal{D}(x)$ (for $1 \leq j \leq |B|$), so that $\text{dep}(x, e_j) \subseteq B_j$ for e_j the j'th element of B (note $\text{dep}(x, e_j) \subseteq B_j$ if $(\text{sat}(x) \setminus \text{supp}(x)) \cap B_j = \emptyset$).

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- But $g \notin B_j$ means $g \notin \text{dep}(x, e_j)$, which means $g \not\preceq e_j$, meaning B is compatible with $\preceq$.

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- Conversely (Compatible ⇒ Generate): Suppose ordering B is compatible with $\preceq$ and that $\text{supp}(x) \subseteq B \subseteq \text{sat}(x)$.

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- Then for each j (with $1 \leq j \leq |B|$), and for each $e \in B_j$, we have $\text{dep}(x, e) \subseteq B_j$.

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- Thus, B_j is the union of tight sets (since each of $\text{dep}(x, e)$ is tight), so that B_j is also tight (unions of tight sets are tight).

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- Thus, B_j is the union of tight sets (since each of $\text{dep}(x, e)$ is tight), so that B_j is also tight (unions of tight sets are tight).
- Thus B is tight and thus x is generated by the ordering given in B (by the greedy algorithm).

□

Extreme Points and Greedy

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If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) \subseteq B \subseteq \text{sat}(x)$, then x is generated using greedy by some ordering of B.

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- this is a more satisfying way to given an extreme point show that the greedy algorithm can generate it than to resort to the polyhedral $cv = max(cx : x \in P_f)$ for an appropriate direction c.
- Moreover, we can produce an efficient $O(|E|^2)$ algorithm that can produce $\preceq$ for any extreme point x of P_f.
- The algorithm does so by, for each $e \in \text{sat}(x)$, producing the sets $\text{dep}(x, e)$ that define the order (or otherwise terminating by saying that x is not an extreme point).

Extreme point testing and partial order generation

input : Vector $x \in \mathbb{R}^E$, polymatroid function f on E.

output: That x is not extreme point, or if it is, minimal tight sets $\text{dep}(x, e)$ for $e \in \text{sat}(x)$ thus defining $\preceq$. Moreover, $\text{dep}(x, e_j) = A_j$ for $1 \leq j \leq n$ where $n = |\text{sat}(x)|$.

$j \leftarrow 0$; $B \leftarrow \emptyset$;
while *true* **do**
  $j \leftarrow j + 1$;
  if $\exists e \in E \setminus B$ *with* $x(B + e) = f(B + e)$ **then**
    $B \leftarrow B + e$; $e_j \leftarrow e$;
  else
    STOP, if $\text{supp}(x) \subseteq B$ then x is extreme, otherwise not.
  $A_j \leftarrow B$; $k \leftarrow j - 1$;
  while $x(A_j - e_k) = f(A_j - e_k)$ *and* $k > 0$ **do**
    $A_j = A_j - e_k$; $k \leftarrow k - 1$;

On partial order algorithm

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- Thus, extreme point testing is fundamentally computationally simpler than arbitrary membership testing (recall, to test if $x \in P_f$ in general, we needed submodular function minimization).
- To determine, only, if a given x is extreme, we can delete lines 8-10 (having same run time).

Maximal in a tight set

Theorem 3.6

Given an extreme point $x \in P_f$, with A tight for x, and if given order $\preceq$ element $e \in A$ is maximal, then $A - e$ is also tight.

Proof.

- If e is maximal in A w.r.t. $\preceq$, then there exists **no** $a \in A \setminus \{e\}$, such that $e \in \text{dep}(x, a)$.

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- Now, since $\text{dep}(x, a)$ is the smallest x-tight set containing a and $\text{dep}(x, a) \subseteq A \setminus \{e\}$, we have

$$\bigcup_{a \in A \setminus \{e\}} \text{dep}(x, a) = A \setminus \{e\} \tag{5}$$

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- Since the union (and intersection) of tight sets is tight, we have that $A \setminus \{e\}$ is therefore also tight.

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Corollary 3.7

For all $e \in \text{sat}(x)$, we have that $\text{dep}(x, e) \setminus e$ is also tight.

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- $\text{dep}(x, e)$ is tight, and recall that there is some ordered set B_j^e with $\text{dep}(x, e) = B_j^e$ whose's last (j'th) item is e.

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Maximal in a tight set

- Also, for any $x \in P_f$, and $\forall e \in \text{sat}(x)$, we have that

$$\text{dep}(x, e) \setminus \{e\} \subseteq \text{supp}(x) \tag{6}$$

- This follows since suppose $\exists e' \in \text{dep}(x, e) \setminus \{e\}$ such that $x(e') = 0$.
- Then, since $f(e') > 0$, in such case $\text{dep}(x, e)$ wouldn't be minimally e-containing tight, since we'd have $x(\text{dep}(x, e) \setminus \{e'\}) = x(\text{dep}(x, e)) = f(\text{dep}(x, e))$.

On Greedy, and linear programming max

Theorem 3.8

Let $y \in P_f$ be an extreme point, and let $\preceq$ be the partial order of y. Let $c \in \mathbb{R}^E$. Then, y is the solution in:

$$c^\intercal y = \max\{c^\intercal x : x \in P_f\} \tag{7}$$

iff the following three conditions hold:

(1) $c(e) \geq 0$ *for every* $e \in \text{supp}(y)$

(2) $c(e) \leq 0$ *for every* $e \in E \setminus \text{sat}(y)$, *and*

(3) *For* $d, e \in \text{sat}(y)$ *and* $d \preceq e$ *imply that* $c(d) \geq c(e)$.

Another revealing theorem

Theorem 3.9

Let f be a polymatroid function and suppose that E can be partitioned into $(E_1, E_2, \ldots, E_k)$ such that $f(A) = \sum_{i=1}^{k} f(A \cap E_i)$ for all $A \subseteq E$, and k is maximum. Then the base polytope $B_f = \{x \in P_f : x(E) = f(E)\}$ (the E-tight subset of P_f) has dimension $|E| - k$.

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- Thus, "independence" between disjoint A and B (leading to a rectangular projection of the polymatroid polytope) reduces the dimension of the base polytope, as expected.

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- Thus, "independence" between disjoint A and B (leading to a rectangular projection of the polymatroid polytope) reduces the dimension of the base polytope, as expected.
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- And if f does not have such independence, dimension of B_f is $|E| - 1$ and any point $x \in B_f$ is a convex combination of at most $|E|$ vertices of B_f.

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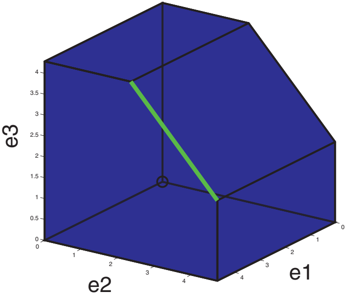
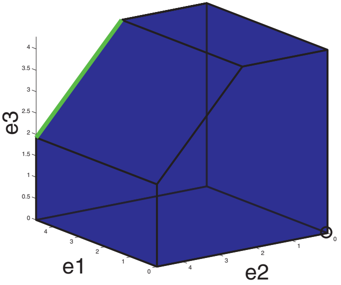
- Example f with independence between $A = \{e_2, e_3\}$ and $B = \{e_1\}$, i.e., $e_1 \perp\!\!\!\perp \{e_2, e_3\}$, with B_f marked in green.





Base polytope existence

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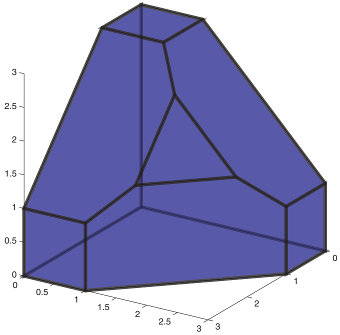
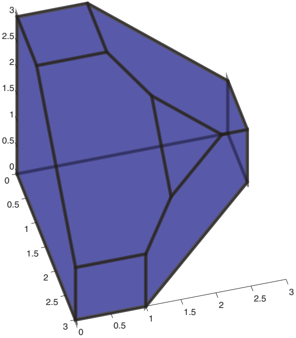
- In words, B_f intersects all "multi-axis orthogonal" subsets of R_+^E.

Polytope example 1

- Observe: P_f (at two views):





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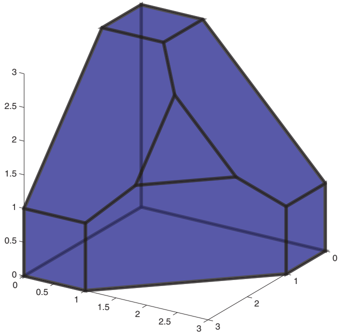

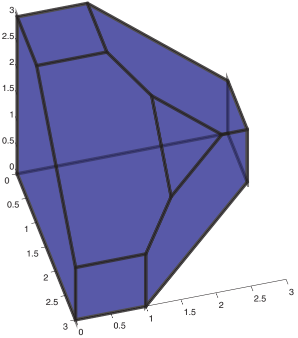

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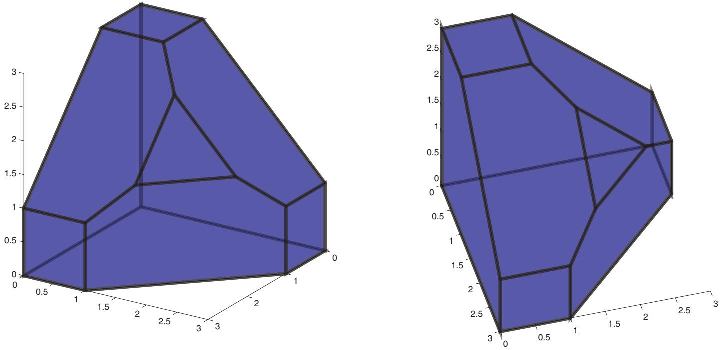

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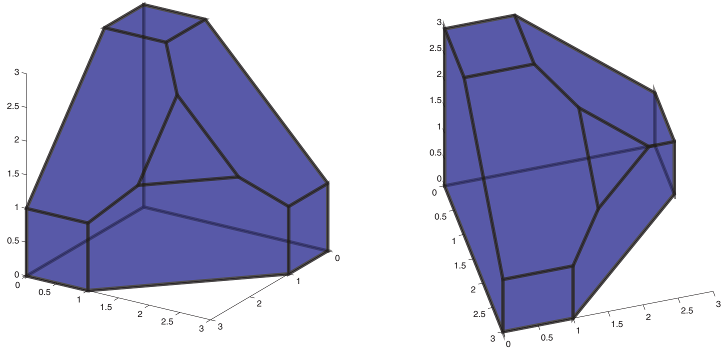

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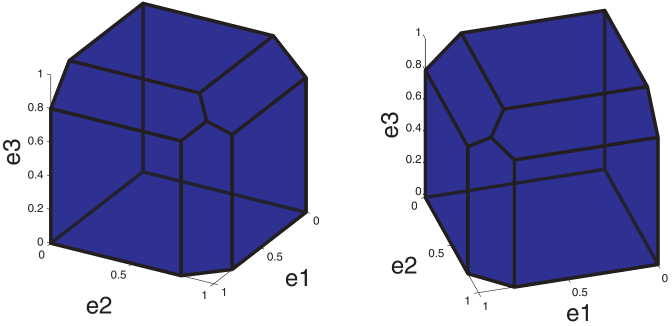

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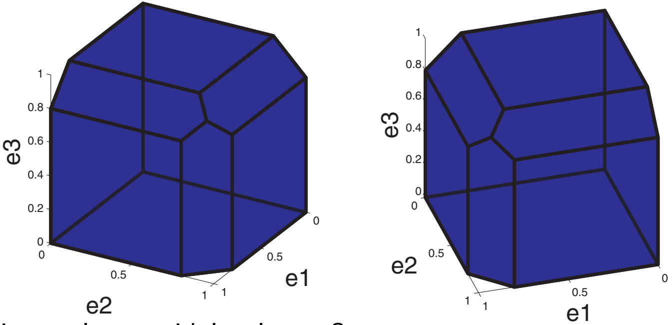

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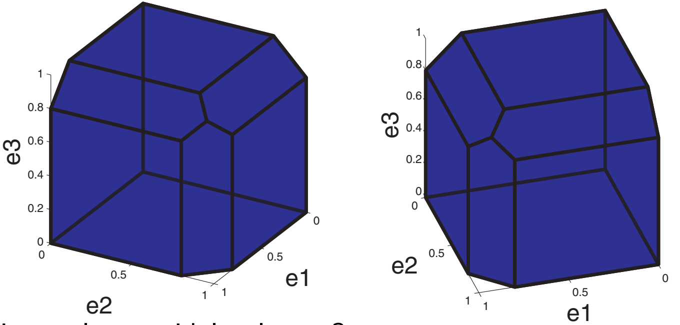

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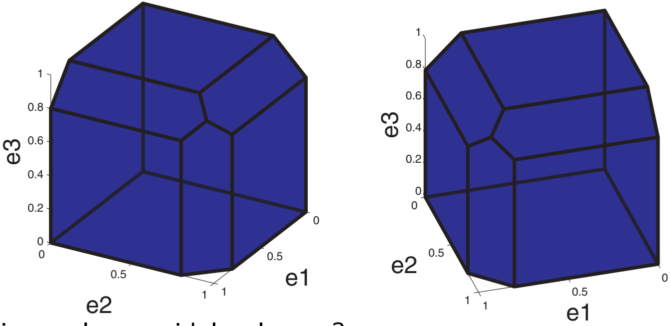

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→ SFM for arbitrary submodular g (from lecture 11)

- First, given any submodular function g, construct modular function $m : E \to \mathbb{R}$ such that $m(e) = g(E \setminus \{e\}) - g(E)$.
  Note that

$$
\begin{aligned}
m(e) &= g(E \setminus \{e\}) - g(E) && (9)\\
&= -[g(E) - g(E \setminus e)] && (10)\\
&= -[\text{ gain of adding } e \text{ to } E \setminus e \text{ }] && (11)\\
&= -[\text{ smallest possible gain/value of } e \text{ in any context }] && (12)
\end{aligned}
$$

The last equality follows from submodularity.

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- Then $f(\emptyset) = 0$, so f is normalized.
- Also, f is monotone non-decreasing (and thus non-negative) and submodular. It is submodular since sum of submodular and modular. Monotone non-decreasing follows since for $v \notin B$,

$$\begin{aligned} f(B + v) - f(B) &= g(B + v) - g(B) + m(v) && (14) \\ &= g(B + v) - g(B) + g(E - v) - g(E) && (15) \\ &\geq 0 && (16) \end{aligned}$$

since, by submodularity, $g(B + v) - g(B) \geq g(E) - g(E - v)$.

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- So now we have a difference of a polymatroid function f and a modular function m.

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- This follows since f is monotone non-decreasing, and $m(A) = m(A') + m(e)$, so $m(A') > m(A)$.
- So we "throw away" any e s.t. $m(e) < 0$. We get a new function on $E' = E \setminus M$ where $M = \{e : m(e) < 0\}$, and define new function $f' : 2^{E'} \to \mathbb{R}_+$ with $f'(A) = f(A)$ for $A \subseteq E'$. This deals with (2) above.

$\rightarrow$ SFM on arbitrary submodular g: transformation

- Given any arbitrary submodular function g with the goal of finding $A^* \in \operatorname{argmin}_{A \subseteq E} g(A)$
- We reduce this to:

$$A^* \in \operatorname*{argmin}_{A \subseteq E'} \Big(f(A) - m(A) \Big) \tag{17}$$

  where

  - f is a polymatroid function on $2^{E'}$
  - m is a modular function on $2^{E'}$ with $m \in \mathbb{R}_+^{E'}$.
  - $E' \subseteq E$.

- In the sequel, we assume this form, with ground set E.
- Moreover, we may assume that P_f is a polymatroidal polytope, with $P_f \subset \mathbb{R}_+^E$.

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- Thus, this can act as a certificate of optimality for any submodular function minimization problem on g even if g is not polymatroidal.
- We need only find a feasible y on the max (left) side, and an A^* on the min (right) side that achieves equality, then A^* is a SFM solution in $A^* \in \operatorname{argmin}_{A \subseteq E} g(A)$ where x is the aforementeioned modular function, and $f(A) = g(A) + m(A) - g(\emptyset)$.

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- At each step of the algorithm, we either find a larger y, or demonstrate y's optimality by finding a minimizing A.

Maximizing y

- The nature of SFM will be very similar to the Edmonds's matroid partition problem (recall, asking if E can be partitioned into $\{I_i\}$ each independent in a matroid M_i) and the core algorithm is very similar.
- Now, from convex polytope theory, any $x \in P_f$ can be represented as a convex combination of at most $|E| + 1$ extreme points of P_f (each of which may be generated by greedy).
- We keep a feasible solution to the max version of the problem as a convex combination of such extreme points.
- That is, let I be an index set, and $x^{(i)}$ be an extreme point of P_f for $i \in I$. We then keep y as

$$y = \sum_{i \in I} \lambda_i x^{(i)} \tag{19}$$

  where λ_i are convex coefficients.
- At each step of the algorithm, we either find a larger y, or demonstrate y's optimality by finding a minimizing A.
- Start with $y = 0$, $I = \{1\}$, $\lambda_1 = 1$, and $v^{(1)} = 0$.

From vertex to vertex

- We will need to move from one extreme point to another (adjacent) extreme point, and will use an augmenting path like approach to do so.
- How do we characterize such adjacent extreme points?

From vertex to vertex

Theorem 4.2

Let x be an extreme point of P_f, and let $\preceq$ be its partial order. Then, each of the following three operations will yield a new extreme point w:

(a) *Let $a, b \in E$ and a cover b relative to $\preceq$. Let $w = x + \alpha \mathbf{1}_a - \alpha \mathbf{1}_b$ with $\alpha = f(\text{dep}(x, a) - b) - x(\text{dep}(x, a) - b)$.*

From vertex to vertex

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From vertex to vertex

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(c) *Let $a \in \text{supp}(x)$ be maximal (w.r.t. $\preceq$), and let $w - x - x(a)\mathbf{1}_a$.*

Scratch Paper

Scratch Paper

Scratch Paper

Sources for Today's Lecture

- Birkhoff, "Lattice Theory", 1967.
- Bixby, Cunningham, Topkis, "The Partial Order of a Polymatroid Extreme Point", 1985.
- J. Edmonds, "Submodular Functions, Matroids, and Certain Polyhedra", 1970.
- Lovász, "Submodular Functions and Convexity", 1983.