EE595A – Submodular functions, their optimization and applications – Spring 2011

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Department of Electrical Engineering
Spring Quarter, 2011
http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 18 - June 1st, 2011

Announcements

- Last lecture, and final presentations, will take place Thursday, June 9th, from 3-7:30pm. The lecture will be from 3:00-5:00pm, and the final presentations will be from 5:00-7:30pm. Please bring dinner.
- Today: short lecture (due to many deadlines this week).
- office hours on wed 12:30pm.

Class Road Map

We need to find one makeup lecture this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L11 (5/6): On SFM, polymatroid member & greedy, Lovász ext.
- L12 (5/11): Lovász ext. + polymatroid props.
- L13 (5/13): More polymatroids, start lattices
- L14 (5/18): lattices/submodular
- L15 (5/20): lattices, → SFM.
- L16 (5/25): → SFM
- L17 (5/27): dep/sat
- L18 (6/1): exchange capacities
- L19 (6/3):
- L20: (6/9): 3-7:30pm (EEB-303)?

dep and partial order

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Theorem 2.1

If $x \in P_f$ *is an extreme point, then* $\preceq$ *is a partial order on* $\text{sat}(x)$ *where for* $a, e \in \text{sat}(x)$*, the order* $\preceq$ *is defined by:* $a \preceq e$ *iff* $a \in \text{dep}(x, e)$*.*

if $x \in P_f$, $\text{sat}(x) \subseteq E$.

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- In fact, we have a stronger result that extreme points are characterized by this construct:

Theorem 2.2

$x \in P_f$ *is an extreme point, iff* $\text{supp}(x) \subseteq \text{sat}(x)$ *and* $\text{dep}(x, a) \neq \text{dep}(x, b)$ *for every pair of distinct points* $a, b \in \text{sat}(x)$*.*

the partial order of extreme points

Theorem 2.3

Let x be an extreme point of P_f and $\preceq$ be its partial order. Let $B \subseteq E$ be an ordered set. Then B generates x using the greedy algorithm iff we have $\text{supp}(x) \subseteq B \subseteq \text{sat}(x)$ *and B is compatible with $\preceq$.*

Corollary 2.4

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) \subseteq B \subseteq \text{sat}(x)$*, then x is generated using greedy by some ordering of B.*

Extreme point testing and partial order generation

```
input : Vector $x \in \mathbb{R}^E$, polymatroid function $f$ on $E$.
output: That $x$ is not extreme point, or if it is, minimal tight sets $\text{dep}(x, e)$
        for $e \in \text{sat}(x)$ thus defining $\preceq$. Moreover, $\text{dep}(x, e_j) = A_j$ for
        $1 \leq j \leq n$ where $n = |\text{sat}(x)|$.
$j \leftarrow 0$ ; $B \leftarrow \emptyset$ ;
while true do
    $j \leftarrow j + 1$ ;
    if $\exists e \in E \setminus B$ with $x(B + e) = f(B + e)$ then
        $B \leftarrow B + e$, $e_j \leftarrow e$.
    else
        STOP, if $\text{supp}(x) \subseteq B$ then $x$ is extreme, otherwise not.
    $A_j \leftarrow B$; $k \leftarrow j - 1$ ;
    while $x(A_j - e_k) = f(A_j - e_k)$ and $k > 0$ do
        $A_j = A_j - e_k$; $k \leftarrow k - 1$
```

On Greedy, and linear programming max

Theorem 2.5

Let $y \in P_f$ *be an extreme point, and let* $\preceq$ *be the partial order of* y. *Let* $c \in \mathbb{R}^E$. *Then,* y *is the solution in:*

$$c^\intercal y = \max\left\{c^\intercal x : x \in P_f\right\} \tag{1}$$

iff the following three conditions hold:

(1) $c(e) \geq 0$ *for every* $e \in \text{supp}(y)$

(2) $c(e) \leq 0$ *for every* $e \in E \setminus \text{sat}(y)$, *and*

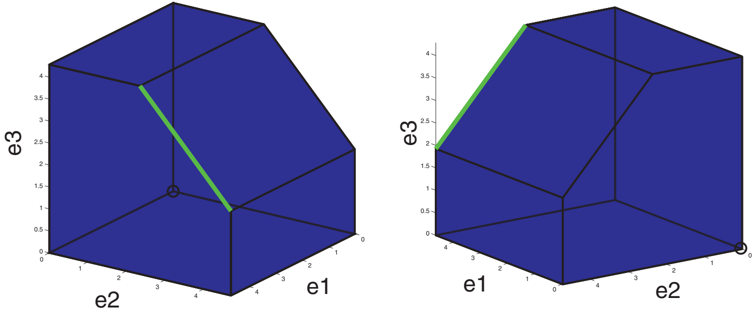
(3) *For* $d, e \in \text{sat}(y)$ *and* $d \preceq e$ *imply that* $c(d) \geq c(e)$.

Another revealing theorem

Theorem 2.6

Let f *be a polymatroid function and suppose that* E *can be partitioned into* $(E_1, E_2, \dots, E_k)$ *such that* $f(A) = \sum_{i=1}^{k} f(A \cap E_i)$ *for all* $A \subseteq E$*, and* k *is maximum. Then the base polytope* $B_f = \{x \in P_f : x(E) = f(E)\}$ *(the* E*-tight subset of* P_f*) has dimension* $|E| - k$*.*

- Example f with independence between $A = \{e_2, e_3\}$ and $B = \{e_1\}$, i.e., $e_1 \perp\!\!\!\perp \{e_2, e_3\}$, with B_f marked in green.



Base polytope existence and location

- Given polymatroid function f, the base polytope $B_f = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A)\ \forall A \subseteq E, \text{ and } x(E) = f(E)\right\}$ always exists.

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$$B_f \cap \left\{x \in \mathbb{R}_+^E : x(A) = f(A)\right\} \neq \emptyset \tag{2}$$

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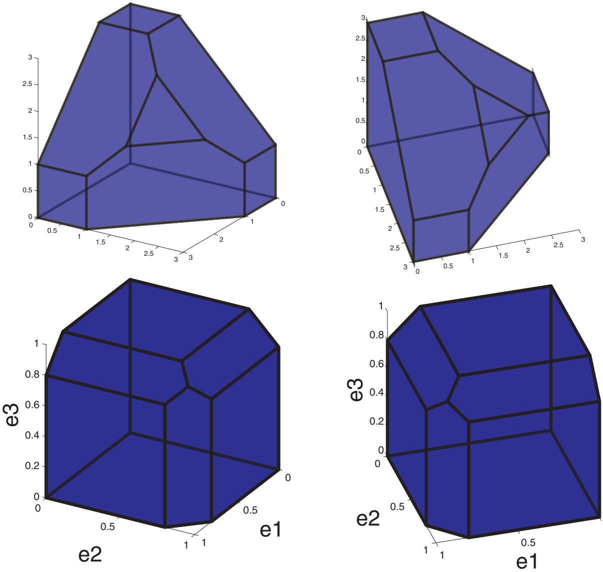
- In words, B_f intersects all "multi-axis orthogonal" subsets of R_+^E.

Not polymatroidal polytopes



$\rightarrow$ SFM on arbitrary submodular g: transformation

- Given any arbitrary submodular function g with the goal of finding $A^* \in \operatorname{argmin}_{A \subseteq E} g(A)$
- We reduce this to:

$$A^* \in \operatorname*{argmin}_{A \subseteq E'} \Big(f(A) - m(A) \Big) \tag{3}$$

  where

  - f is a polymatroid function on $2^{E'}$
  - m is a modular function on $2^{E'}$ with $m \in \mathbb{R}_+^{E'}$.
  - $E' \subseteq E$.

- In the sequel, we assume this form, with ground set E.
- Moreover, we may assume that P_f is a polymatroidal polytope, with $P_f \subset \mathbb{R}_+^E$.

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- Thus, this can act as a certificate of optimality for any submodular function minimization problem on g even if g is not polymatroidal.
- We need only find a feasible y on the max (left) side, and an A^* on the min (right) side that achieves equality, then A^* is a SFM solution in $A^* \in \operatorname{argmin}_{A \subseteq E} g(A)$ where x is the aforementioned modular function, and $f(A) = g(A) + m(A) - g(\emptyset)$.

Maximizing y

- The nature of SFM will be very similar to the Edmonds's matroid partition problem (recall, asking if E can be partitioned into $\{I_i\}$ each independent in a matroid M_i) and the core algorithm is very similar.

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- That is, let I be an index set, and $x^{(i)}$ be an extreme point of P_f for $i \in I$. We then keep y as

$$y = \sum_{i \in I} \lambda_i x^{(i)} \tag{5}$$

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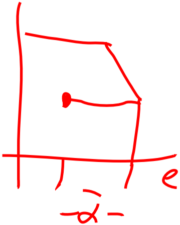
- Start with $y = 0$, $I = \{1\}$, $\lambda_1 = 1$, and $v^{(1)} = 0$.

Saturation Capacity

- For $x \in P_f$, and $e \in E$, consider finding

$$\max\{\alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f\} = \tilde{\alpha} \tag{6}$$



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- Identical to:
$$\max\{\alpha : (x + \alpha \mathbf{1}_e)(A) \leq f(A), \forall A \supseteq \{e\}\} \tag{7}$$
since $B \subseteq E$ such that $e \notin B$ have the same value $(x + \alpha \mathbf{1}_e)(B) = x(B)$.

$1_e(B) = 0$ if $e \notin B$.

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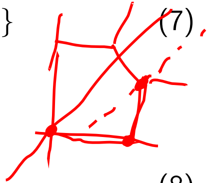

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- This max is achieved when
$$\alpha = \hat{c}(x; e) \overset{\text{def}}{=} \min \{f(A) - x(A), \forall A \supseteq \{e\}\} \tag{10}$$
- $\hat{c}(x; e)$ is known as the saturation capacity associated with $x \in P_f$ and e.

Saturation Capacity

- Thus we have for $x \in P_f$,

$$\hat{c}(x;e) \overset{\text{def}}{=} \min \{f(A) - x(A), \forall A \supseteq \{e\}\} \tag{11}$$

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- Note that any α with $0 \leq \alpha \leq \hat{c}(x;e)$ we have $x + \alpha \mathbf{1}_e \in P_f$.
- We also see that computing $\hat{c}(x;e)$ is a form of submodular function minimization.

hc?

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both e and e'

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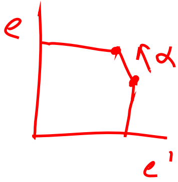

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$A : e \in A, e' \notin A$

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- Which is identical to:

$$\max \left\{\alpha : \alpha \in \mathbb{R}, \alpha(\mathbf{1}_e - \underbrace{\mathbf{1}_{e'}))(A)} \le f(A) - x(A), \forall A \supseteq \{e\}, e' \notin A\right\} \tag{16}$$

Exchange Capacity

- In such case, we get $\mathbf{1}_{e'}(A) = 0$, thus above identical to

$$\max \left\{ \alpha : \alpha \in \mathbb{R}, \alpha \mathbf{1}_e(A) \leq f(A) - x(A), \forall A \supseteq \{e\}, e' \notin A \right\} \tag{17}$$

any e' s.t. $e' \notin A$, $I_{e'}(A) = 0$.

any e s.t. $e \in A$, $I_e(A) = 1$

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$$\alpha = \hat{c}(x; e, e') \overset{\text{def}}{=} \min\left\{f(A) - x(A), \forall A \supseteq \{e\}, e' \notin A\right\} \tag{19}$$

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- $\hat{c}(x; e, e')$ is known as the exchange capacity associated with $x \in P_f$ and e.

- if e & e' are successive in order that generates extreme point x, then we get other extreme point
$$x' = x + \hat{c}(x; e, e')(\mathbf{1}_e - \mathbf{1}_{e'})$$

- note also, (19) is also a form of SFM

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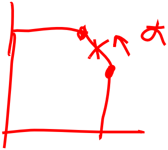
- For any α with $0 \leq \alpha \leq \hat{c}(x; e, e')$, we have that $x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f$.



dep revisited

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dep revisited

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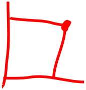
- Perhaps, then, a better name for dry is nsat(x), for the necessary for tightness. (aka nsat)

for this x, $\text{dry}(x) = \text{sat}(x) = \emptyset$



$\text{sat}(x) = \{e_1, e_2\}$

$\text{dry}(x) = \emptyset$

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dep revisited

- Now, given $x \in P_f$, and $e \in \text{sat}(x)$, recall distributive lattice of e-containing tight sets $\mathcal{D}(x, e) = \{A : e \in A, x(A) = f(A)\}$

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- We can see $\text{dry}(x, e)$ as the elements that are necessary for e-containing tightness, with $e \in \text{sat}(x)$.
- That is, we can view $\text{dry}(x, e)$ as

$$\text{dep}(x,e) = \text{dry}(x, e) = \left\{ e' : x(A) < f(A), \forall A \not\ni e', e \in A \right\} \tag{21}$$

- This can be read as, for any $e' \in \text{dry}(x, e)$, any e-containing set that does not contain e' is not tight for x.
- Notice also that $\text{dry}(x, e) = \text{dep}(x, e)$.

dep revisited

- Now, we have the following equalities for $\text{dep}(x, e)$:

$$
\begin{aligned}
\text{dep}(x, e) &= \{e' : x(A) < f(A), \forall A \not\ni e', e \in A\} && (22)\\
&= \{e' : \exists \alpha > 0,\ \text{s.t. } \alpha \leq f(A) - x(A), \forall A \not\ni e', e \in A\} && (23)\\
&= \{e' : \exists \alpha > 0,\ \text{s.t. } \alpha \mathbf{1}_e(A) \leq f(A) - x(A), \forall A \not\ni e', e \in A\} && (24)\\
&= \{e' : \exists \alpha > 0,\ \text{s.t. } \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A) - x(A), \forall A \not\ni e', e \in A\} && (25)\\
&= \{e' : \exists \alpha > 0,\ \text{s.t. } x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A), \forall A \not\ni e', e \in A\} && (26)
\end{aligned}
$$

dep revisited

- Now, we have the following equalities for dep(x, e):

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\end{aligned}$$

- Now, $1_e(A) - \mathbf{1}_{e'}(A) = 0$ if either $\{e, e'\} \subseteq A$, or $\{e, e'\} \cap A = \emptyset$.

dep revisited

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\end{aligned}$$

- Now, $1_e(A) - \mathbf{1}_{e'}(A) = 0$ if either $\{e, e'\} \subseteq A$, or $\{e, e'\} \cap A = \emptyset$.
- Also, if $e' \in A$ but $e \notin A$, then $x(A) + \alpha(\underbrace{\mathbf{1}_e(A)}_{0} - \underbrace{\mathbf{1}_{e'}(A)}_{1}) = x(A) - \alpha \le f(A)$ since $x \in P_f$.

trivially true since

if $x \in P_f$, $x(A) - \alpha \le f(A)$ $\forall A$

$\alpha > 0$

α small

dep revisited

- thus, we get the same in the above if we remove the constraint $A \not\ni e', e \in A$, that is we get

$$\text{dep}(x, e) = \left\{ e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A), \forall A \right\} \quad (27)$$

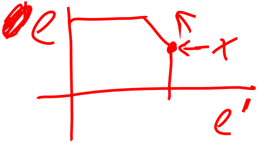
- This is then identical to

$$\text{dep}(x, e) = \left\{ e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f \right\} \quad (28)$$

note,

$e \in \text{dep}(x,e)$

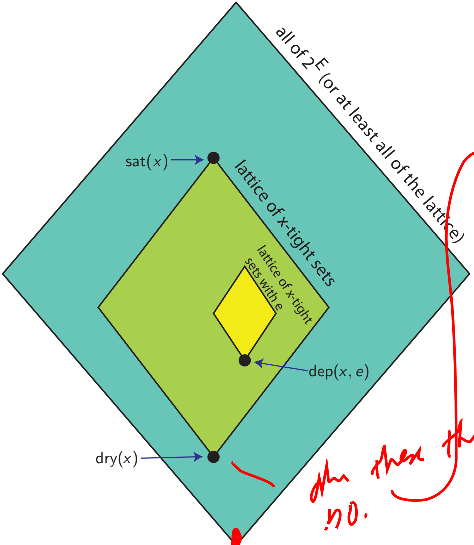
since $\mathbf{1}_e - \mathbf{1}_e = 0$

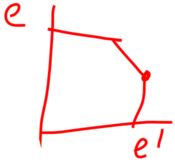


$\text{dep}(x,e) \setminus \{e\} = \{e'\}$

dep and sat

The following picture summarizes the relationships.





$e' \cap e, e' = e'$

are these the same points?

no.

From vertex to vertex

- We will need to move from one extreme point to another (adjacent) extreme point, and will use an augmenting path like approach to do so.
- How do we characterize such adjacent extreme points?

From vertex to vertex

Theorem 3.1

Let x be an extreme point of P_f, and let $\preceq$ be its partial order. Then, each of the following three operations will yield a new extreme point w:

(a) *Let $a, b \in E$ and a cover b relative to $\preceq$, so $b \sqsubset a$. Let $w = x + \alpha \mathbf{1}_a - \alpha \mathbf{1}_b$ with $\alpha = f(\mathrm{dep}(x, a) - b) - x(\mathrm{dep}(x, a) - b)$.*

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(b) *Let $a \in E \setminus \text{sat}(x)$, and let $w = x + \alpha \mathbf{1}_a$ where $\alpha = f(\text{sat}(x) + a) - f(\text{sat}(x))$.*

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(b) *Let $a \in E \setminus \text{sat}(x)$, and let $w = x + \alpha \mathbf{1}_a$ where $\alpha = f(\text{sat}(x) + a) - f(\text{sat}(x))$.*

(c) *Let $a \in \text{supp}(x)$ be maximal (w.r.t. $\preceq$), and let $w = x - x(a)\mathbf{1}_a$.*

From Vertex to Vertex

- For (a), let x be generated by $E_i = (e_1, e_2, \ldots, e_{k-1}, b, a, e_{k+2}, \ldots, e_i)$ and consider generating w with an order with a and b swapped, i.e., $E'_i = (e_1, e_2, \ldots, e_{k-1}, a, b, e_{k+2}, \ldots, e_i)$

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- Then

$$x(e_k) = f(\text{[illegible]} (e_1, \ldots, e_{k-1}) \tag{29}$$

$$x(e_{k+1}) = f(e_1, . \text{[illegible]} a) - f(e_1, \ldots, e_{k-1}, b) \tag{30}$$

$$w(e_k) = f(e_1 \text{[illegible]} - f(e_1, \ldots, e_{k-1}) \tag{31}$$

$$w(e_{k+1}) = f(\text{[illegible]} k-1 \text{[illegible]} f(e_1, \ldots, e_{k-1}, a) \tag{32}$$

From Vertex to Vertex

- For (a), let x be generated by $E_i = (e_1, e_2, \dots, e_{k-1}, b, a, e_{k+2}, \dots, e_i)$ and consider generating w with an order with a and b swapped, i.e., $E_i' = (e_1, e_2, \dots, e_{k-1},$ [illegible] $b, e_{k+2}, \dots, e$ [illegible]
- Then

$$x(e_k) = \text{[illegible]} - f(e_1, \dots, e_{k-1}) \quad (29)$$

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$$w(e_{k+1}) = \text{[illegible]}, e_k \text{[illegible]} - f(e_1, \dots, e_{k-1}, a) \quad (32)$$

- Also, $(w - x)(e) = 0$ [illegible] ll $e \notin \{e_k,$ [illegible] $\}$ and

$$(w - x)(e_k) = f(e_1, \dots, e_{k-1}, a) - f(e_1, \dots, e_k, b) \quad (33)$$

$$(w - x)(e_{k+1}) = f(e_1, \dots, e_{k-1}, b) - f(e_1, \dots, e_k, a) \quad (34)$$

$(w-x)(e_{k+n}) = f($ ~~[illegible]~~ $e_{1, k-1}, a, b, e_{k+2})$

$-$ same $= 0$

From Vertex to Vertex

- For (a), let x be generated by $E_i = (e_1, e_2, \ldots, e_{k-1}, b, a, e_{k+2}, \ldots, e_i)$ and consider generating w with an order with a and b swapped, i.e., $E_i' = (e_1, e_2, \ldots, e_{k-1}, a, b, e_{k+2}, \ldots, e_i)$
- Then

$$x(e_k) = f(e \text{[illegible]} {}_{-1}, b) - \text{[illegible]}, e_{k-1}) \tag{29}$$

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- Also, $(w - x)(e) = 0$ for a[illegible], e_k[illegible]

$$(w - x)(e_k) = \text{[illegible]} \ldots, e_{k-1}, \text{[illegible]} (e_1, \ldots, e_k, b) \tag{33}$$

$$(w - x)(e_{k+1}) = f(e_1, \ldots, e_{k-1}, b) - f(e_1, \ldots, e_k, a) \tag{34}$$

- So with $\alpha = f(e_1, \ldots, e_{k-1}, a) - f(e_1, \ldots, e_k, b)$ we have

$$w = x + \alpha(\mathbf{1}_a - \mathbf{1}_b) \tag{35}$$

B_f dominates

Lemma 3.2

Let $x \in P_f$ *and let* $T = \text{sat}(x)$*. Then there exists* $y \in B_f$ *such that* $y \geq x$ *with* $y(e) = x(e)$ *for* $e \in T$*.*

Proof.

- Consider a form of the greedy procedure, where we update x

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• $e \notin \text{sat}$

- Thus, after x update, e, we still have $x \in P_f$.
- Moreover, at each update there is a set S_e that achieves the min in the min form of $c(x; e)$. This set S_e is tight for the new x and remains tight for all subsequent iterations.

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- Thus, after x update, e, we still have $x \in P_f$.
- Moreover, at each update there is a set S_e that achieves the min in the min form of $c(x; e)$. This set S_e is tight for the new x and remains tight for all subsequent iterations.
- Eventually we stop, and since $E = T \cup \bigcup_{e \notin T} S_e$ is the union of tight sets (for x), we see that the resulting x has $x \in B_f$.

Scratch Paper

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Sources for Today's Lecture

- Bixby, Cunningham, Topkis, "The Partial Order of a Polymatroid Extreme Point", 1985.
- J. Edmonds, "Submodular Functions, Matroids, and Certain Polyhedra", 1970.
- Lovász, "Submodular Functions and Convexity", 1983.