EE595A – Submodular functions, their optimization and applications – Spring 2011

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering
Spring Quarter, 2011
http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 19 - June 3st, 2011

Announcements

- Last lecture, and final presentations, will take place Thursday, June 9th, from 3-7:30pm in room EEB-303.
- The lecture will be from 3:00-5:00pm,
- The final presentations will be from 5:00-7:30pm. Feel free to bring dinner.
- Final project reports due next Wednesday, June 8th, at 11:45pm (on the web page, dropbox will be posted shortly).
- Final slides due by 1:00pm on Thursday, also via the dropbox.
- Final talks: Plan for 10 minutes of talking about your project (perhaps 8 minutes of lecture and 2 minutes of questions).

Class Road Map

We need to find one makeup lecture this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L11 (5/6): On SFM, polymatroid member & greedy, Lovász ext.
- L12 (5/11): Lovász ext. + polymatroid props.
- L13 (5/13): More polymatroids, start lattices
- L14 (5/18): lattices/submodular
- L15 (5/20): lattices, → SFM.
- L16 (5/25): → SFM
- L17 (5/27): dep/sat
- L18 (6/1): exchange capacities
- L19 (6/3): SFM algorithm
- L20: (6/9): 3-7:30pm (EEB-303)?

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- That is, let I be an index set, and $x^{(i)}$ be an extreme point of P_f for $i \in I$. We then keep y as

$$y = \sum_{i \in I} \lambda_i x^{(i)} \tag{1}$$

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where λ_i are convex coefficients.
- At each step of the algorithm, we either find a larger y, or demonstrate y's optimality by finding a minimizing A.
- Start with $y = 0$, $I = \{1\}$, $\lambda_1 = 1$, and $v^{(1)} = 0$.

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- For $x \in P_f$, and $e \in E$, consider finding

$$\max \{\alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f\} \tag{2}$$

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since $B \subseteq E$ such that $e \notin B$ have the same value $(x + \alpha \mathbf{1}_e)(B) = x(B)$.

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- Note that any α with $0 \leq \alpha \leq \hat{c}(x; e)$ we have $x + \alpha \mathbf{1}_e \in P_f$.
- We also see that computing $\hat{c}(x; e)$ is a form of submodular function minimization.

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- In such case, we get $\mathbf{1}_{e'}(A) = 0$, thus above identical to

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- $\hat{c}(x; e, e')$ is known as the exchange capacity associated with $x \in P_f$ and e.
- For any α with $0 \leq \alpha \leq \hat{c}(x; e, e')$, we have that $x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f$.

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- This can be read as, for any $e' \in \text{dry}(x)$, any set that does not contain e' is not tight for x.

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$$\text{dry}(x, e) = \left\{e' : x(A) < f(A), \forall A \not\ni e', e \in A\right\} \tag{17}$$

dep revisited

- Now, given $x \in P_f$, and $e \in \text{sat}(x)$, recall distributive lattice of e-containing tight sets $\mathcal{D}(x, e) = \{A : e \in A, x(A) = f(A)\}$
- We can define the "1" element of this lattice as $\text{sat}(x, e) \stackrel{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x, e)\}$.
- Analogously, we can define the "0" element of this lattice as $\text{dry}(x, e) \stackrel{\text{def}}{=} \bigcap \{A : A \in \mathcal{D}(x, e)\}$.
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- That is, we can view $\text{dry}(x, e)$ as

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- This can be read as, for any $e' \in \text{dry}(x, e)$, any e-containing set that does not contain e' is not tight for x.
- Notice also that $\text{dry}(x, e) = \text{dep}(x, e)$.

dep revisited

- Now, we have the following equalities for $\text{dep}(x, e)$:

$$\text{dep}(x, e) = \{e' : x(A) < f(A), \forall A \not\ni e', e \in A\} \tag{18}$$

$$= \{e' : \exists \alpha > 0, \text{ s.t. } \alpha \leq f(A) - x(A), \forall A \not\ni e', e \in A\} \tag{19}$$

$$= \{e' : \exists \alpha > 0, \text{ s.t. } \alpha \mathbf{1}_e(A) \leq f(A) - x(A), \forall A \not\ni e', e \in A\} \tag{20}$$

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$$= \{e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A), \forall A \not\ni e', e \in A\} \tag{22}$$

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- Now, $1_e(A) - \mathbf{1}_{e'}(A) = 0$ if either $\{e, e'\} \subseteq A$, or $\{e, e'\} \cap A = \emptyset$.

dep revisited

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- Now, $1_e(A) - \mathbf{1}_{e'}(A) = 0$ if either $\{e, e'\} \subseteq A$, or $\{e, e'\} \cap A = \emptyset$.
- Also, if $e' \in A$ but $e \notin A$, then $x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) = x(A) - \alpha \leq f(A)$ since $x \in P_f$.

dep revisited

- thus, we get the same in the above if we remove the constraint $A \not\ni e', e \in A$, that is we get

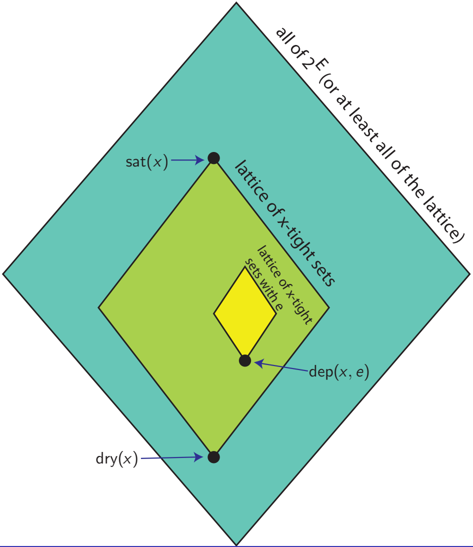
$$\text{dep}(x, e) = \left\{e' : \exists \alpha > 0,\ \text{s.t. } x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A), \forall A\right\} \quad (23)$$

- This is then identical to

$$\text{dep}(x, e) = \left\{e' : \exists \alpha > 0,\ \text{s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\right\} \quad (24)$$

dep and sat

The following picture summarizes the relationships.



From vertex to vertex

- We will need to move from one extreme point to another (adjacent) extreme point, and will use an augmenting path like approach to do so.
- How do we characterize such adjacent extreme points?

From vertex to vertex

Theorem 3.1

Let x be an extreme point of P_f, and let $\preceq$ be its partial order. Then, each of the following three operations will yield a new extreme point w:

(a) *Let $a, b \in E$ and a cover b relative to $\preceq$, so $b \sqsubset a$. Let $w = x + \alpha \mathbf{1}_a - \alpha \mathbf{1}_b$ with $\alpha = f(\text{dep}(x, a) - b) - x(\text{dep}(x, a) - b)$.*

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(b) *Let $a \in E \setminus \text{sat}(x)$, and let $w = x + \alpha \mathbf{1}_a$ where $\alpha = f(\text{sat}(x) + a) - f(\text{sat}(x))$.*

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(b) *Let $a \in E \setminus \text{sat}(x)$, and let $w = x + \alpha \mathbf{1}_a$ where* $\alpha = f(\text{sat}(x) + a) - f(\text{sat}(x))$.

(c) *Let $a \in \text{supp}(x)$ be maximal (w.r.t. $\preceq$), and let $w = x - x(a)\mathbf{1}_a$.*

From Vertex to Vertex

- For (a), let x be generated by $E_i = (e_1, e_2, \ldots, e_{k-1}, b, a, e_{k+2}, \ldots, e_i)$ and consider generating w with an order with a and b swapped, i.e., $E_i' = (e_1, e_2, \ldots, e_{k-1}, a, b, e_{k+2}, \ldots, e_i)$

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- Then

$$x(b) = f(e_1, \ldots, e_{k-1}, b) - f(e_1, \ldots, e_{k-1}) \tag{25}$$

$$x(a) = f(e_1, \ldots, e_{k-1}, b, a) - f(e_1, \ldots, e_{k-1}, b) \tag{26}$$

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- First, we have $(w - x)(e) = 0$ for all $e \notin \{a, b\}$ since in each case the differences are the same.

From Vertex to Vertex

- For a we have:

$$(w - x)(a) = f(e_1, \dots, e_{k-1}, a) - f(e_1, \dots, e_{k-1}) \tag{29}$$

$$- f(e_1, \dots, e_{k-1}, a, b) + f(e_1, \dots, e_{k-1}, b) \tag{30}$$

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\end{aligned}
$$

- For b, we have

$$
\begin{aligned}
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$$(w-x)(a) = -(w-x)(b) \tag{33}$$

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- so we immediately see that

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- So with $\alpha = (w-x)(a)$ we have

$$w = x + \alpha(\mathbf{1}_a - \mathbf{1}_b) \tag{34}$$

From Vertex to Vertex

- Moreover, we see that since

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$$f(e_1,\dots,e_{k-1},a) - f(e_1,\dots,e_{k-1}) \geq f(e_1,\dots,e_{k-1},a,b) - f(e_1,\dots,e_{k-1},b) \tag{38}$$

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- We have that

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- Now both x and w are extreme points (generated by greedy algorithm), so any point $\lambda x + (1-\lambda)w$ is in the base of $\text{supp}(x)$
- If all E is used, then each of x and w are in B_f and the convex combination lies on one of the skeletal edges of B_f.

From Vertex to Vertex and exchange capacity

- Therefore, defining

$$\alpha = (w - x)(a) = f(e_1, \ldots, e_{k-1}, a) - f(e_1, \ldots, e_{k-1}) \tag{40}$$
$$- f(e_1, \ldots, e_{k-1}, a, b) + f(e_1, \ldots, e_{k-1}, b) \tag{41}$$
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we see that $\hat{c}(x; a, b) \geq \alpha$

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- On the other hand, moving from x to w jumps between two extreme points, and if we moved any less than $\hat{c}(x; a, b)$ this wouldn't be the case.

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- Thus,

$$\alpha = \hat{c}(x; a, b) \stackrel{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{a\}, b \notin A \} \tag{43}$$

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- Interestingly, this particular submodular minimization problem is easy.

B_f dominates

Lemma 3.2

Let $x \in P_f$ and let $T = \text{sat}(x)$. Then there exists $y \in B_f$ such that $y \geq x$ with $y(e) = x(e)$ for $e \in T$.

Proof.

- Consider a form of the greedy procedure, where we update x

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$$x \leftarrow x + \hat{c}(x; e) \tag{44}$$

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- Iterate the following procedure, for any $e \notin \text{sat}(x)$

$$x \leftarrow x + \hat{c}(x; e)\mathbf{1}_e \tag{44}$$

- Thus, after x update, e, we still have $x \in P_f$.

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- Moreover, at each update there is a set S_e that achieves the min in the min form of $\hat{c}(x; e)$. This set S_e is tight for the new x and remains tight for all subsequent iterations.

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- Consider a form of the greedy procedure, where we update x
- Iterate the following procedure, for any $e \notin \text{sat}(x)$
$$x \leftarrow x + \hat{c}(x; e) \tag{44}$$
- Thus, after x update, e, we still have $x \in P_f$.
- Moreover, at each update there is a set S_e that achieves the min in the min form of $c(x; e)$. This set S_e is tight for the new x and remains tight for all subsequent iterations.
- Eventually we stop, and since $E = T \cup \bigcup_{e \notin T} S_e$ is the union of tight sets (for x), we see that the resulting x has $x \in B_f$.

More on dep

Lemma 3.3

Given any extreme point $x \in P_f$, then for any $e \in \text{supp}(x)$, we have $x(e) = f(\text{dep}(x, e)) - f(\text{dep}(x, e) \setminus e)$ $= \rho_e(\text{dep}(x,e) \setminus e)$

Proof.

- Consider all orderings that generate x and choose the one where e occurs earliest.

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- Let $E = (e_1, \dots, e_i)$ be that order, letting $e = e_k$.

e_k

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Proof.

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- Let $E = (e_1, \dots, e_i)$ be that order, letting $e = e_k$.
- Then $\text{dep}(x, e) = \{e_1, \dots, e_k\}$
- Since this generates x, we have $x(e) = f(x_1, \dots, x_k) - f(x_1, \dots, x_{k-1}) = f(\text{dep}(x, e)) - f(\text{dep}(x, e) \setminus e)$

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- Then $\operatorname{dep}(x,e) = \{e_1, \dots, e_k\}$
- Since this generates x, we have $x(e) = f(x_1, \dots, x_k) - f(x_1, \dots, x_{k-1}) = f(\operatorname{dep}(x,e)) - f(\operatorname{dep}(x,e) \setminus e)$
- Alt proof: We saw earlier that both $\operatorname{dep}(x,e)$ and $\operatorname{dep}(x,e) \setminus e$ are tight, since e is maximal in $\operatorname{dep}(x,e)$ using order $\preceq$ on x.

More on dep

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Given any extreme point $x \in P_f$, then for any $e \in \text{supp}(x)$, we have $x(e) = f(\text{dep}(x,e)) - f(\text{dep}(x,e) \setminus e)$

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- This will be used to produce edges from a to b in order to "augment" y via an exchange like operation.

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SFM

Theorem 3.4

If there is an s, t dipath in the graph, then there exists a $y' \in P_f$ with $y \leq y' \leq x$ with $y'(E) > y(E)$. If there is no such dipath, then $y(A) = f(A)$ and $y(E \setminus A) = x(E \setminus A)$ for some $A \subseteq E$.

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- Therefore, y' is a convex combination of at most $|I| + m$ extreme points as given in Equation 49.
- This can then be reduced in $O(n^3)$ time to at most $n+1$ extreme points using Carathéodory's theorem and associated linear algebra routines.

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- Moreover, $\{b : b \preceq_j a\} \subseteq A$ since otherwise there is a $e \in A$ and $d \in E \setminus A$ such that e covers d in $\preceq_j$.

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- But $\{b : b \preceq_j a\}$ is x^j-tight.

SFM

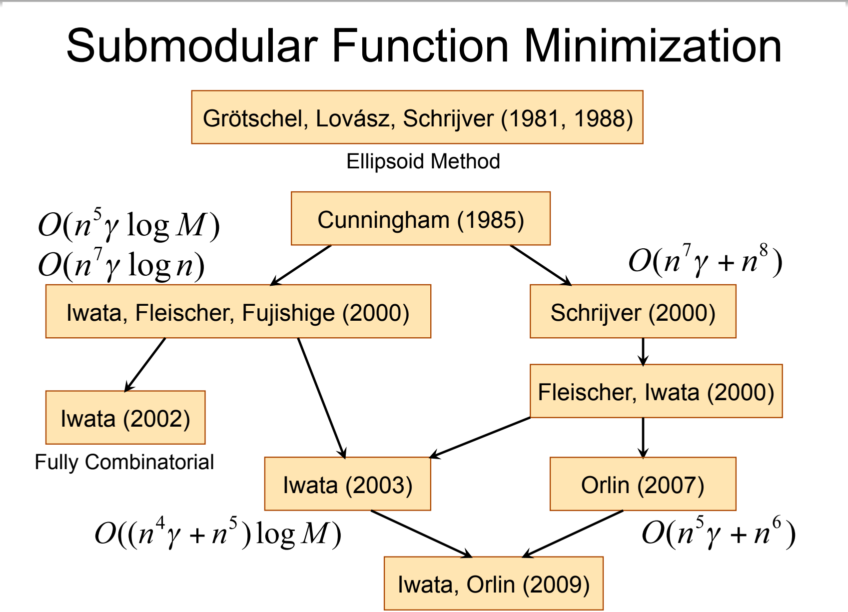
- Next we show that such an A is tight.
- If $a \in A$ and $j \in I$, then $a \in \text{sat}(x^j)$ since (a, t) is not an edge.
- Moreover, $\{b : b \preceq_j a\} \subseteq A$ since otherwise there is a $e \in A$ and $d \in E \setminus A$ such that e covers d in $\preceq_j$.
- But $\{b : b \preceq_j a\}$ is x^j-tight.
- Therefore, A is the union of x^j-tight sets and so itself is x^j-tight, as required.

SFM

Theorem 3.5

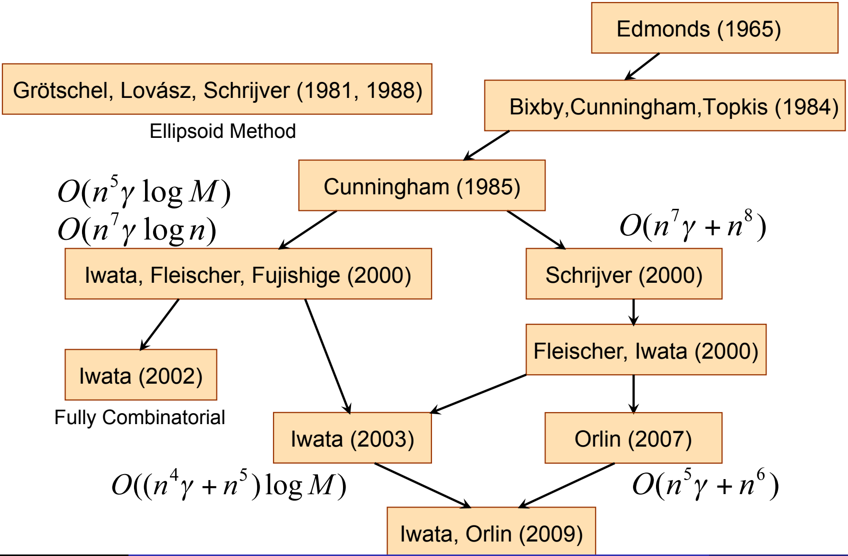
If f is integer valued, then so is x, and in such case it is possible to choose the steps δ in such a way that the running time of the algorithm is $O(Mn^6 \log(Mn)\beta)$ where M is an integer bound on the max value of f, $n = |E|$, and β is the cost of evaluating a submodular function query.

SFM Summary (from S. Iwata's slides on SFM)



SFM Summary (from S. Iwata's slides on SFM)

Submodular Function Minimization



Announcements

- Last lecture, and final presentations, will take place Thursday, June 9th, from 3-7:30pm in room EEB-303.
- The lecture will be from 3:00-5:00pm,
- The final presentations will be from 5:00-7:30pm. Feel free to bring dinner.
- Final project reports due next Wednesday, June 8th, at 11:45pm (on the web page, dropbox will be posted shortly).
- Final slides due by 1:00pm on Thursday, also via the dropbox.
- Final talks: Plan for 10 minutes of talking about your project (perhaps 8 minutes of lecture and 2 minutes of questions).

Scratch Paper

Scratch Paper

Scratch Paper

Sources for Today's Lecture

- Cunningham, "On Submodular Function Minimization", 1985.
- Bixby, Cunningham, Topkis, "The Partial Order of a Polymatroid Extreme Point", 1985.
- J. Edmonds, "Submodular Functions, Matroids, and Certain Polyhedra", 1970.
- Lovász, "Submodular Functions and Convexity", 1983.