EE595A – Submodular functions, their optimization and applications – Spring 2011

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http://ee.washington.edu/class/235/2011wtr/index.html

Lecture 4 - April 8th, 2011

Announcements

- Goal is to have HW1 ready this weekend (so please look out for it).

Matroid

Definition 2.1 (Matroid)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down closed)

(I3') $\forall I, J \in \mathcal{I}$, with $|I| > |J|$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$

Matroids

In fact, we can use the rank of a matroid for its definition.

Theorem 2.2 (Matroid from rank)

Let E be a set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \;\; 0 \leq r(A) \leq |A|$ *(non-negative cardinality bounded)*

(R2) $r(A) \leq r(B)$ *whenever* $A \subseteq B \subseteq E$ *(monotone non-decreasing)*

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ *forall* $A, B \subseteq E$ *(submodular)*

- So submodular non-negative integral monotone non-decreasing cardinality bounded is necessary and sufficient to define the matroid.

Matroids

We also saw that it is possible to uniquely define a matroid based on either:

- Independence
- Rank axioms
- Base axioms
- Circuit axioms
- Closure axioms (we didn't see this yesterday, but it is possible)

Matroid, examples

Examples of matroids include

- Matric matroids (characterized by linear independence)
- Graphic matroids (cycle matroid of a graph)
- "free" matroid (all subsets of E)
- k-uniform matroid (all subsets of size at most k)

Partition Matroid

- Let V be our ground set.
- Let $V = V_1 \cup V_2 \cup \dots \cup V_\ell$ be a partition of V into disjoint sets (disjoint union). Define a set of subsets of V as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \dots, \ell\}. \tag{1}$$

  where $k_1, \dots, k_\ell$ are fixed parameters. Then $M = (V, \mathcal{I})$ is a matroid. $k_i \geq 0$
- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- We'll show that property (I3') in Def 2.1 holds. If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Partition Matroid

- A partition matroid has rank function? What is it?

Partition Matroid

- A partition matroid has rank function

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \tag{2}$$

which we also immediately see is submodular using properties we spoke about last week.

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  - $\min(\text{submodular}(A), k_i)$ is submodular in A since $|A \cap V_i|$ is monotone.

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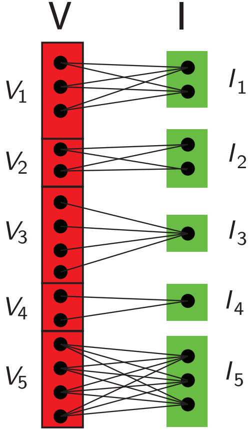
    - sums of submodular functions are submodular.
- $r(A)$ is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).

Partition Matroid, rank as neighbor cardinality

- A partition matroid can be viewed using a bipartite graph.
- Letting V denote the ground set, and $V_1, V_2, \dots$ the partition, the graph is $G = (V, I, E)$ where V is the ground set, I is a set of "indices", and E is the set of edges.
- $I = (I_1, I_2, \dots, I_\ell)$ is a set of $\sum_{i=1}^{\ell} k_i$ nodes, grouped into ℓ clusters, where there are k_i nodes in the i^{th} group I_i.
- $(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$.

Partition Matroid, rank as neighbor cardinality

- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 3, 1, 1, 3)$.



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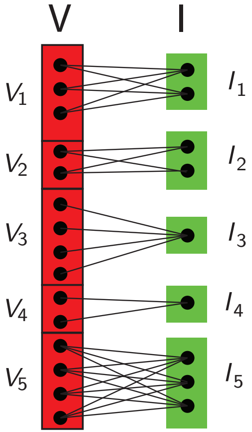

- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and how $|\Gamma(X)|$ is submodular.

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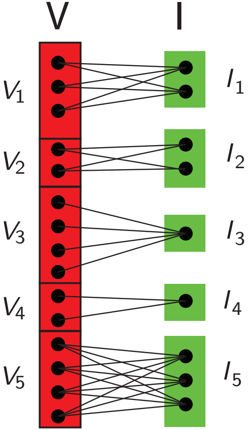

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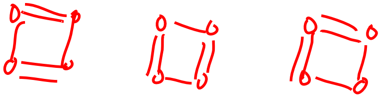

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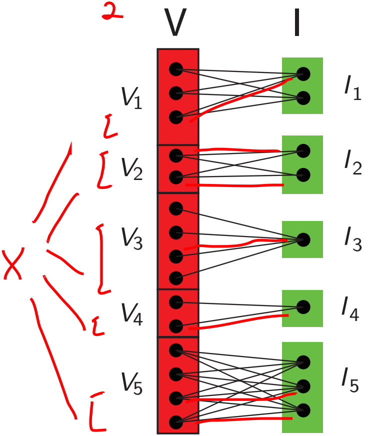

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- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a construted bi-partite graph, the rank function of a partition matroid is $r(X) = |\Gamma(X)| = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i)$.

= max matching out of X

System of Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_k : i \in I)$ where $V_i \subseteq V$ for all i).

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  $|I| \leq |V|$ V

- A family $(v_i : i \in I)$ for index set I is said to be a system of representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \to I$ such that $v_i \in V_{\pi(i)}$

  v_i is the representative of set $\pi(i)$

  the ith representation is meant to represent $\pi(i)$

  v_i = John i = Miling County

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T must be distinct.

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System of Representatives

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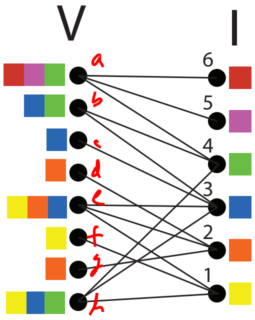
V might be a multiset

$\ell = 6 \quad \mathcal{V} = \{ \{e f h\}_{V_1}, \{d g\}_{V_2}, \{b c e h\}_{V_3}, \{a b h\}_{V_4}, \{a\}_{V_5}, \{a\}_{V_6} \}$



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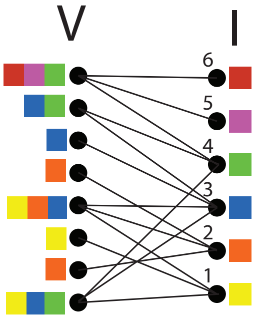

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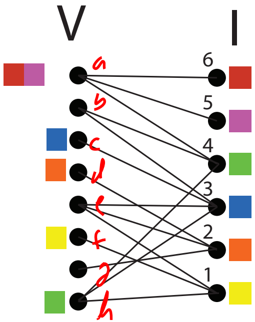

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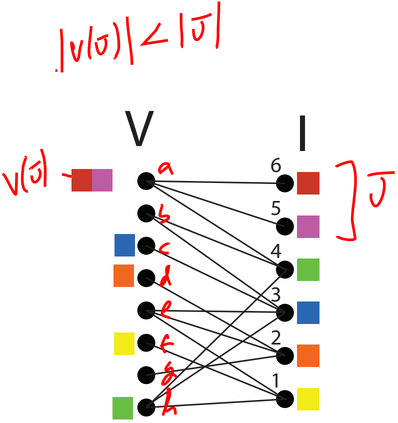

- A system of representatives would make sure that there is a representative for each color group. For example,
- The representatives are shown as colors on the left.
- Note that in this example, the set of representatives is not distinct. In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).

System of Distinct Representatives

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Definition 4.1 (transversal)

Given a set system $(V, \mathcal{V})$, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \tag{3}$$

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- Note that due to it being a bijection, all of I and T are "covered" (so this makes things distinct).

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- Therefore, for any transversal T, any subset $T' \subseteq T$ is a partial transversal (down closed).

When Transversals?

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- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, for any $J \subseteq I$, let

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- We have

Theorem 5.1 (Hall's theorem)

Given a set system $(V, \mathcal{V})$, the family of subsets $\mathcal{V} = (V_i : i \in I)$ has a transversal iff for all $J \subseteq I$

$$|V(J)| \geq |J| \tag{5}$$

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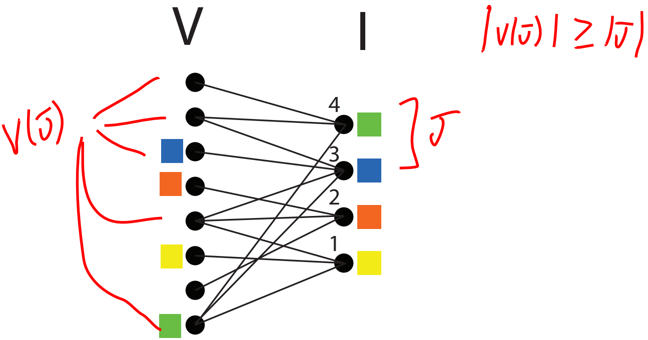
- Hall's theorem as a bipartite graph.



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- Moreover, we have

Theorem 5.2 (Rado's theorem)

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal which is independent in M iff for all $J \subseteq I$
$$r(V(J)) \geq |J| \tag{6}$$

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- Note, a transversal T independent in M means that $r(T') = |T'|$ for all $T' \subseteq T$.

When Transversals?

Theorem 5.3

If $\mathcal{V} = (V_i : I \in I)$ *is a finite family of non-empty subsets of* V*, and* $f : 2^V \to \mathbb{Z}_+$ *is a non-negative, integral, monotone non-decreasing, and submodular function, then* $\mathcal{V}$ *has a system of representatives* $(v_i : i \in I)$ *such that*

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{8}$$

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- Given Theorem 5.3, we immediately get Theorem ~~3.2~~ 5.1 by taking $f(S) = |S|$ for $S \subseteq V$. *In which case, Eq. 7 requires the system of representatives to be distinct.*

When Transversals?

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- Given Theorem 5.3, we immediately get Theorem 3.2 by taking $f(S) = |S|$ for $S \subseteq V$.
- We get Theorem 3.3 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 7 insists the system of representatives is indepedent in M.*

When Transversals?

$$V(J) \supseteq \bigcup_{i \in J} \{v_i\}$$

$$\Rightarrow f(V(J)) \geq f\Big(\bigcup_{i \in J} \{v_i\}\Big)$$

first part proof of Theorem 5.3.

- Suppose Eq. 7 is true. Then since f is monotone, and since $V(J) \supseteq \cup_{i \in J}\{v_i\}$ when $(v_i : i \in I)$ is a system of representatives, then Eq. 8 immediately follows.

...

$$V(J) = \bigcup_{j \in J} V_j = V_1 \cup V_2 \cup \dots \cup V_{|J|}$$

$$f(V(J)) \geq |J|$$

$$\text{is } f(V_1 \setminus \{f\} \cup V_2 \cup V_3 \cup \dots \cup V_{|J|}) \overset{?}{\geq} |J|$$

When Transversals?

Lemma 5.4

Suppose Eq. 8 is true for $\mathcal{V}$, and there exists an i such that $|V_i| > 2$. W.l.o.g. let $i = 1$. Then there exists $f \in V_1$ such that the family of subsets $(V_1 \setminus \{f\}, V_2, \ldots, V_n)$ also satisfies Eq 8.

Proof.

$f(V(J)) \geq |J| \quad \forall J \subseteq I$

- When Eq. 8 holds, this means that for any subsets $J_1, J_2 \subseteq \{2, \ldots, n\}$, we have that

$$f(V_1 \cup V(J_1)) \geq |J_1| + 1 \tag{9}$$

$$f(V_1 \cup V(J_2)) \geq |J_2| + 1 \tag{10}$$

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Proof.

- Suppose, to the contrary, this is false. Then we may take f_1, f_2 as two distinct elements in V_1,
- and there must exist subsets J_1, J_2 of $\{2, \dots, n\}$ such that

$$f((V_1 \setminus \{f_1\}) \cup V(J_1)) < |J_1| + 1, \quad (11)$$

$$f((V_1 \setminus \{f_2\}) \cup V(J_2)) < |J_2| + 1, \quad (12)$$

(note that either one or both of J_1, J_2 could be empty).

...

violating eq. 8.

$< |J_1| + 1 \equiv \leq |J_1|$

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Proof.

- Taking $X = (V_1 \setminus \{f_1\}) \cup V(J_1)$ and $Y = (V_1 \setminus \{f_2\}) \cup V(J_2)$, we see that:

$$X \cup Y = V_1 \cup V(J_1 \cup J_2) \tag{13}$$

and

$$X \cap Y \supseteq V(J_1 \cap J_2) \tag{14}$$

. . .

When Transversals?

Lemma 5.4

Suppose Eq. 8 is true for $\mathcal{V}$, and there exists an i such that $|V_i| > 2$. W.l.o.g. let $i = 1$. Then there exists $f \in V_1$ such that the family of subsets $(V_1 \setminus \{f\}, V_2, \ldots, V_n)$ also satisfies Eq 8.

Proof.

- since f is submodular and monotone non-decreasing, we get

$$|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \tag{15}$$

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- Since $\mathcal{V}$ satisfies Eq. 8, and $1 \notin J_1 \cup J_2$, this gives
$$|J_1| + |J_2| \geq |J_1 \cup J_2| + 1 + |J_1 \cap J_2| \tag{16}$$
which is a contradiction.

□

When Transversals?

converse proof of Theorem 5.3.

- Conversely, suppose Eq. 8 is true.

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This theorem can be used to produce a variety of other results quite easily, and shows how submodularity is the key ingredient in its truth.

Transversal Matroid

Transversals, themselves, define a matroid.

Theorem 6.1

If $\mathcal{V}$ is a family of finite subsets of a ground set V, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on V.

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- This means that the transversals of $\mathcal{V}$ are the bases of matroid M. Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!

Transversals and Matchings

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- Given a set system $(V, \mathcal{V})$, with $\mathcal{V} = (V_i : i \in I)$, we can define a bipartite graph $G = (V, I, E)$ associated with $\mathcal{V}$ that has edge set $\{(v, i) : v \in V, i \in I, v \in I_i\}$.

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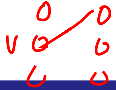
- A matching in this graph is a set of edges no two of which that have a common endpoint.
- In fact, we easily have



Lemma 6.2

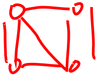
A subset $T \subseteq V$ is a partial transversal of $\mathcal{V}$ iff there is a matching in (V, I, E) in which every edge has one endpoint in V.

We say that T is matched into I.

Morphing Partition Matroid Rank

- Recall the partition matroid rank function. Note also that $k_i = |I_i|$ in the bipartite graph representation.

Are matchings in arbitrary graphs matroids?



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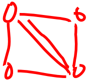

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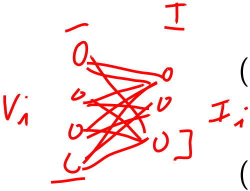

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$$= \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap A| + |I_i \setminus J_i| \right) \tag{20}$$

... Morphing Partition Matroid Rank

- Continuing,

$$r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i| \right) \tag{21}$$

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- In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.

Partial Transversals Are Matroids

In fact, we have

Theorem 6.3

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \ldots, V_\ell)$ be a subset system. Let $I = \{1, \ldots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

Proof.

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- Suppose that T_1 and T_2 are partial transversals of $\mathcal{V}$ such that $|T_1| < |T_2|$. Exercise: show that (I3') holds.

□

Transversal Matroid Rank

- Transversal matroid has rank

$$r(A) = \min_{J \subseteq I} \left(|V(J) \cap A| - |J| + |I| \right) \tag{25}$$

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Transversal Matroid Rank

$V(\mathcal{I}_i)$

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- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions. Is this true in general? Exercise:

$$r(A) = \min_{J \subseteq I} m_J(A)$$

Matroid and the greedy algorithm

- Let $\mathcal{I}$ be a set of subsets of E that is down-closed. Consider a modular weight function $w : E \to \mathbb{R}$, and we want to find the $A \in \mathcal{I}$ that maximizes $w(A)$.
- Greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that $A \cup \{y\} \in \mathcal{I}$ with $w(y)$ as large as possible, stopping when no such y exists.

Theorem 7.1

Let $\mathcal{I}$ be a non-empty collection of subsets of a set E, down-closed. Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Scratch Paper

Scratch Paper

Scratch Paper

Sources for Today's Lecture

Korte,Vygen-2005, Vondrak-2010, Schrijver-2003, Oxley-1992, Welsh-1973, Goemans-2010.