

EE595A – Submodular functions, their optimization and applications – Spring 2011

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http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 5 - April 13th, 2011

Announcements

- HW1 is on the web page now, at http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/hw1.pdf
- It is due, Tuesday, April 26th, 11:45pm
- All submissions must be done electronically, via our drop box. See the link
<https://catalyst.uw.edu/collectit/dropbox/bilmes/14888>, or look at the homework on the web page.

Matroid

Definition 2.1 (Matroid)

A set system (E, \mathcal{I}) is a **Matroid** if

$$(I1') \quad \emptyset \in \mathcal{I}$$

$$(I2') \quad \forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \text{ (down-closed)}$$

$$(I3') \quad \forall I, J \in \mathcal{I}, \text{ with } |I| > |J|, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I}$$

Matroids

In fact, we can use the rank of a matroid for its definition.

Theorem 2.2 (Matroid from rank)

Let E be a set and let $r : 2^E \rightarrow \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A, B \subseteq E$:

- (R1) $\forall A \subseteq E \quad 0 \leq r(A) \leq |A|$ (non-negative cardinality bounded)
- (R2) $r(A) \leq r(B)$ whenever $A \subseteq B \subseteq E$ (monotone non-decreasing)
- (R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for all $A, B \subseteq E$ (submodular)

- So submodular non-negative integral monotone non-decreasing cardinality bounded is necessary and sufficient to define the matroid.

System of Distinct Representatives

- Let (V, \mathcal{V}) be a set system (i.e., $\mathcal{V} = (V_k : i \in I)$ where $V_i \subseteq V$ for all i).
- A family $(v_i : i \in I)$ for index set I is said to be a **system of distinct representatives** of \mathcal{V} if \exists a bijection $\pi : I \leftrightarrow I$ such that $v_i \in V_{\pi(i)}$ and $v_i \neq v_j$ for all $i \neq j$.
- In a system of distinct representatives, there **is** a requirement for the representatives to be distinct.

Definition 2.3 (transversal)

Given a set system (V, \mathcal{V}) , a set $T \subseteq V$ is a **transversal** of \mathcal{V} if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \quad (1)$$

-
- Note that due to it being a bijection, all of I and T are “covered” (so this makes things distinct).

Transversal Matroid Rank

- Transversal matroid has rank

$$r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \quad (2)$$

- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions. Is this true in general?

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- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions. Is this true in general? **Exercise:**

Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is **circuit** if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

Matroid loops

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- Such an $\{a\}$ is called a **loop**.
- In a matric (i.e., linear) matroid, the only such loop is the value $\mathbf{0}$, as all non-zero vectors have rank 1.
- Note, we also say that two elements s, t are said to be **parallel** if $\{s, t\}$ is a circuit.

Representable

Definition 3.1

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are **isomorphic** if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

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Definition 3.2

linear matroids on a field Let \mathbf{X} be an $n \times m$ matrix and $E = \{1, \dots, m\}$, where $\mathbf{X}_{ij} \in \mathbb{F}$ for some field, and let \mathcal{I} be the set of subsets of E such that the columns of \mathbf{X} are linearly independent over \mathbb{F} .

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- We can more generally define matroids on a field.

Definition 3.3

Any matroid isomorphic to a linear matroid on a field is called **representable over \mathbb{F}**

Representability of Transversal Matroids

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- In particular:

Theorem 3.4

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.

Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 3.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where \mathcal{I} is all subsets of V of cardinality ≤ 2 except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

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- It can be shown that this is a matroid and is representable.

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- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.

Dual of a Matroid

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- Recall, in cycle matroid of a graph, a spanning set of G is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).

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- Recall, in cycle matroid of a graph, a spanning set of G is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).
- Since the smallest spanning sets are bases, we see that the bases of M are complements of the bases of M^* .

Dual of a Matroid

Theorem 4.1

Let M^ be defined as on previous slide. Then M^* is a matroid.*

Proof.

- Clearly $\emptyset \in I^*$, so (I1') holds.

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Proof.

- Clearly $\emptyset \in \mathcal{I}^*$, so (I1') holds.
- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in M , so must $V \setminus I$. Therefore, (I2') holds.

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- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$.

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- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in M , we can find a base B' of M s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.

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- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in M , we can find a base B' of M s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.
- Since B and J are disjoint, we have both: 1) $B \setminus I$ and $J \setminus I$ are disjoint; and 2) $B \cap I \subseteq I \setminus J$. Also note, B' and I are disjoint.

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Proof.

- Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

$$|B| = |B \cap I| + |B \setminus I| \tag{4}$$

$$\leq |I \setminus J| + |B \setminus I| \tag{5}$$

$$< |J \setminus I| + |B \setminus I| \leq |B'| \tag{6}$$

which is a contradiction. *The last inequality on the right follows since $J \setminus I \subseteq B'$, and $B \setminus I \subseteq B'$ implies that $J \setminus I \cup B \setminus I \subseteq B'$, but since J and B are disjoint, we have that $|J \setminus I| + |B \setminus I| \leq |B'|$.*

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- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.

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which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.
- So B' is disjoint with $I \cup \{v\}$, meaning $B' \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in M , and therefore $I \cup \{v\} \in \mathcal{I}^*$.



Dual Matroid Rank

Theorem 4.2

The rank function r_{M^*} of the dual matroid M^* may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (7)$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. *i.e.*, $|X|$ is modular, complement $f(V \setminus X)$ is submodular if f is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.

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- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$. *The right inequality follows since r_M is submodular.*

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$$|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).$$
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- Monotone non-decreasing follows since, as X increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, r_{M^*} is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.

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Proof.

A set X is independent in (V, r_{M^*}) if and only if

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$$r_M(V \setminus X) = r_M(V) \quad (9)$$

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But a subset X is independent in M^* only if $V \setminus X$ is spanning in M (by the definition of the dual matroid). \square

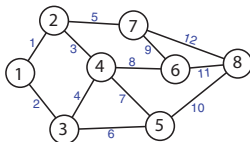
Example duality: cocycle matroid

- The dual of the cycle matroid is called the cocycle matroid.
- It consists of all sets of edges the complement of which contains a spanning tree.

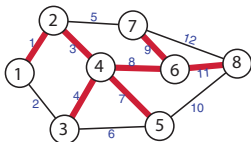
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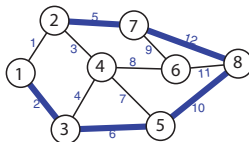
A graph G



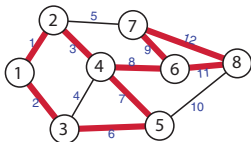
Minimally spanning in M (and thus a base)



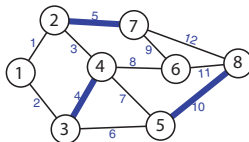
Minimally spanning in M^* (and thus a base)



Spanning in M



Independent in M^*



Matroid and the greedy algorithm

- Let \mathcal{I} be a set of subsets of E that is down-closed. Consider a non-negative modular weight function $w : E \rightarrow \mathbb{R}_+$, and we want to find the $A \in \mathcal{I}$ that maximizes $w(A)$.
- Greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that $A \cup \{y\} \in \mathcal{I}$ with $w(y)$ as large as possible, stopping when no such y exists.

Theorem 5.1

Let \mathcal{I} be a non-empty collection of subsets of a set E , down-closed (i.e., an independence system). Then the pair (E, \mathcal{I}) is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Recall: Matroids by bases

Theorem 5.2

Matroid (by bases) Let E be a set and \mathcal{B} be a nonempty collection of subsets of E . Then the following are equivalent.

- 1 \mathcal{B} is the collection of bases of a matroid;
- 2 if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
- 3 If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”

Matroid and the greedy algorithm

proof of Theorem 5.1.

- Assume (E, \mathcal{I}) is a matroid and $w : E \rightarrow \mathcal{R}_+$ is given.

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- Assume (E, \mathcal{I}) is a matroid and $w : E \rightarrow \mathcal{R}_+$ is given.
- Let $A = (a_1, a_2, \dots, a_r)$ be the solution returned by greedy, where $r = r(M)$ the rank of the matroid, and we order the elements as they were chosen (so $w(a_1) \geq w(a_2) \geq \dots \geq w(a_r)$).

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- A is a base of M , and let $B = (b_1, \dots, b_r)$ be any another base of M with elements also ordered decreasing by weight.

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- A is a base of M , and let $B = (b_1, \dots, b_r)$ be any another base of M with elements also ordered decreasing by weight.
- We next show that not only is $w(A) \geq w(B)$ but that $w(a_i) \geq w(b_i)$ for all i .

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- Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.

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- Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.
- But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen b_i rather than a_k , contradicting what greedy does.



Matroid and the greedy algorithm

converse proof of Theorem 5.1.

- Given an independence system (E, \mathcal{I}) , suppose the greedy algorithm leads to an independent set of max weight for each such weight function. We'll show (E, \mathcal{I}) is a matroid.

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- Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$.
- Define the following modular weight function w on V , and define $k = |I|$.

$$w(v) = \begin{cases} k + 2 & \text{if } v \in I, \\ k + 1 & \text{if } v \in J \setminus I, \\ 0 & \text{if } v \in S \setminus (I \cup J) \end{cases} \quad (10)$$

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- Therefore, (E, \mathcal{I}) must be a matroid.



Matroid restriction/deletion

- Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\} \quad (12)$$

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

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- The rank function is of the same form. I.e., $r_Y : 2^Y \rightarrow \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$.

Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. **Contracting** Z is written M/Z .

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 (see equations 57-60 from lecture 2).
- A **minor** of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.

Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.

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Theorem 6.1

Let M_1 and M_2 be given as above, with rank functions r_1 and r_2 . then the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

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In general, this is an instance of the convolution of two submodular functions, which more generally is written as:

$$(r_1 * r_2)(Y) = \min_{X \subseteq Y} (r_1(X) + r_2(Y \setminus X)) \quad (16)$$

Matroid Union

Definition 6.2

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, \dots , $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the **union** of matroids as

$M_1 \vee M_2 \vee \dots \vee M_k = (V_1 \cup V_2 \cup \dots \cup V_k, \mathcal{I}_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k)$, where

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Theorem 6.3

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, \dots , $M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions r_1, \dots, r_k . Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} (|Y \setminus X| + r_1(X \cap V_1) + \dots + r_k(X \cap V_k)) \quad (18)$$

for any $Y \subseteq V_1 \cup \dots \cup V_k$.

Scratch Paper

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Sources for Today's Lecture

Korte, Vygen-2005, Vondrak-2010, Schrijver-2003, Oxley-1992,
Welsh-1973, Goemans-2010, Piff, Welsh-1970.