EE595A – Submodular functions, their optimization and applications – Spring 2011

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering
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http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 6 - April 15th, 2011

Announcements

- Reminder, HW1 is on the web page now, at http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/hw1.pdf
- It is due, Tuesday, April 26th, 11:45pm
- All submissions must be done electronically, via our drop box. See the link https://catalyst.uw.edu/collectit/dropbox/bilmes/14888, or look at the homework on the web page.

Matroid and the greedy algorithm

- Let $\mathcal{I}$ be a set of subsets of E that is down-closed. Consider a non-negative modular weight function $w : E \to \mathbb{R}_+$, and we want to find the $A \in \mathcal{I}$ that maximizes $w(A)$.
- Greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that $A \cup \{y\} \in \mathcal{I}$ with $w(y)$ as large as possible, stopping when no such y exists.

Theorem 2.1

Let $\mathcal{I}$ be a non-empty collection of subsets of a set E, down-closed (i.e., an independence system). Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Matroid and greedy

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- We don't need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base.
- If we stop at a negative value, we'll once again get a maximum weight independent set.
- We can instead do as small as possible thus giving us a minimum weight independent set/base.

Convex Polyhedra

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Definition 3.1

A subset $P \subseteq \mathbb{R}^E$ is a polyhedron if there exists an $m \times n$ matrix A and vector $b \in \mathbb{R}^E$ (for some $m \geq 0$) such that

$$P = \{x : Ax \leq b\} \tag{1}$$

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- Thus, P is intersection of finitely many affine halfspaces, which are of the form $a_i x \leq b_i$ where a_i is a row vector and b_i a real scalar.

Convex Polytope

- A polytope is defined as follows

Definition 3.2

A subset $P \subseteq \mathbb{R}^E$ is a polytope if it is the convex hull of finitely many vectors in $\mathcal{R}^E$. That is, if $\exists$, $x_1, x_2, \dots, x_k \in \mathcal{R}^E$ such that for all $x \in P$, there exits $\{\lambda_i\}$ with $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0 \; \forall i$ with $x = \sum_i \lambda_i x_i$.

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- We define the convex hull operator as follows:

$$\text{conv}(x_1, x_2, \dots, x_k) \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{k} \lambda_i x_i : \forall i, \; \lambda_i \geq 0, \text{ and } \sum_i \lambda_i = 1 \right\} \tag{2}$$

Convex Polytope - key representation theorem

- A polytope can be defined in a number of ways, two of which include

Theorem 3.3

A subset $P \subseteq \mathbb{R}^E$ is a polytope iff it can be described in either of the following (equivalent) ways:

- *P is the convex hull of a finite set of points.*
- *If it is a bounded intersection of halfspaces, that is there exits matrix A and vector b such that*

$$P = \{x : Ax \leq b\} \tag{3}$$

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- This result follows directly from results proven by Fourier, Motzkin, Farkas, and Carátheodory.

Linear Programming

Theorem 3.4 (weak duality)

Let A be a matrix and b and c vectors, then

$$\max \{c^\intercal x | Ax \leq b\} \leq \min \{y^\intercal b : y \geq 0, y^\intercal A = c^\intercal\} \tag{4}$$

Linear Programming

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Theorem 3.5 (strong duality)

Let A be a matrix and b and c vectors, then

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Linear Programming duality forms

There are many ways to construct the dual. For example,

$$\max\{c^\intercal x | x \geq 0, Ax \leq b\} = \min\{y^\intercal b | y \geq 0, y^\intercal A \geq c^\intercal\} \tag{6}$$

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$$\min\{c^\intercal x | Ax \geq b\} = \max\{t^\intercal b | y \geq 0, y^\intercal A = c^\intercal\} \tag{9}$$

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> *Intuitively, why is [one set of equations] the dual of [another quite different set of equations]? In our experience, this is not the right question to be asked. As stated in Section 12.1, there is a purely mechanical procedure for obtaining the dual of a linear program. Once the dual is obtained, one can devise intuitive, and possibly physical meaningful, ways of thinking about it. Using this mechanical procedure, one can obtain the dual of a complex linear program in a fairly straightforward manner. Indeed, the LP-duality-based approach derives its wide applicability from this fact.*

Vector, modular, incidence

- Recall, any vector $x \in \mathbb{R}^E$ can be seen as a modular function, as for any $A \subseteq E$, we have

$$x(A) = \sum_{a \in A} x_a \tag{10}$$

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- Given an $A \subseteq E$, define the the incidence vector $\mathbf{1}_A \in \{0,1\}^E$ on the unit hypercube as follows:

$$\mathbf{1}_A \stackrel{\text{def}}{=} \left\{ x \in \{0,1\}^E : x_i = 1 \text{ iff } i \in A \right\} \tag{11}$$

equivalently,

$$\mathbf{1}_A(j) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } j \in A \\ 0 & \text{if } j \notin A \end{cases} \tag{12}$$

Matroid

Matroid definition once again.

Definition 4.1 (Matroid)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed)

(I3') $\forall I, J \in \mathcal{I}$, with $|I| > |J|$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$

Independence Polyhedra

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- Now take the rank function r of M, and define the following polyhedron:

$$P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\} \tag{14}$$

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- First note, take any $x \in P_{\text{ind. set}}$, then we have that $x \in P_r$ (or $P_{\text{ind. set}} \subseteq P_r$).

Matroid

- If $x \in P_{\text{ind. set}}$, then
$$x = \sum_i \lambda_i \mathbf{1}_I \tag{15}$$
for some appropriate vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

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$$= \max_{j: I_j \subseteq A} \mathbf{1}_{I_j}(E) \tag{18}$$
$$= r(A) \tag{19}$$
- Thus, $x \in P_r$.

Matroid Polyhedron in 2D

$$P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\} \tag{20}$$

- Consider this in two dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \tag{21}$$

$$x_1 \leq r(\{v_1\}) \tag{22}$$

$$x_2 \leq r(\{v_2\}) \tag{23}$$

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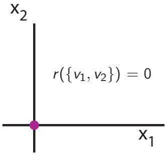
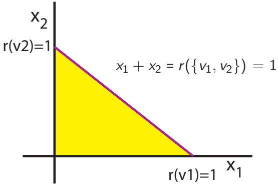
- Because r is submodular, we have

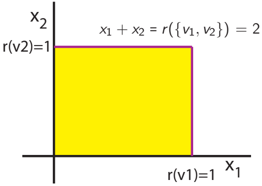
$$r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \tag{25}$$

so since $r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\})$, the last inequality is either touching or active.

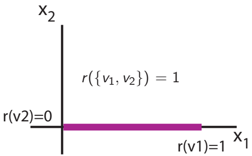
Matroid Polyhedron in 2D

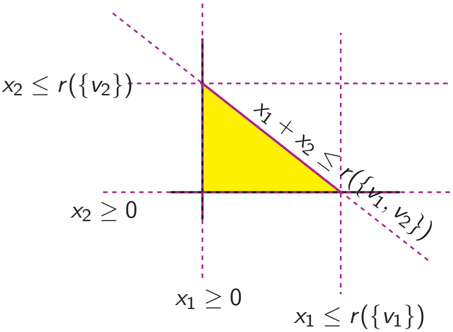






And, if v2 is a loop ...





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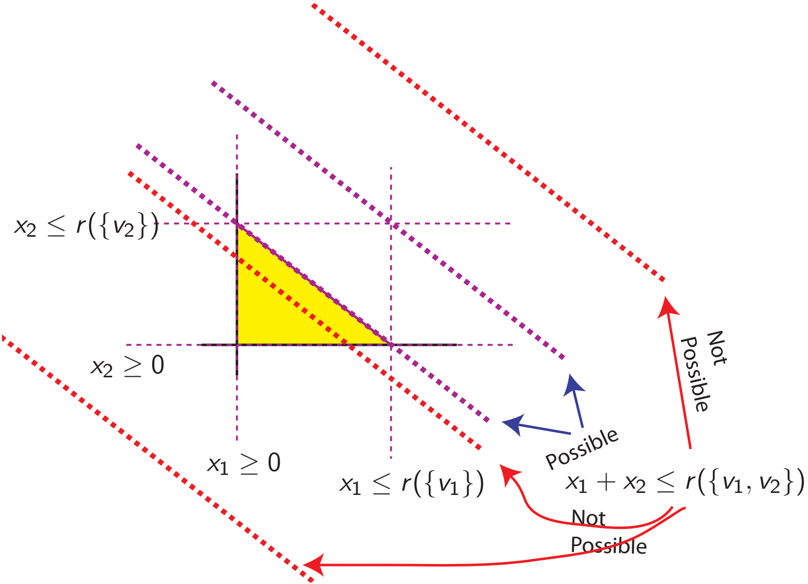

Matroid Polyhedron in 3D

$$P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\} \tag{26}$$

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$$x_2 + x_3 \leq r(\{v_2, v_3\}) \tag{32}$$
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$$x_1 + x_2 + x_3 \leq r(\{v_1, v_2, v_3\}) \tag{34}$$

Matroid Polyhedron in 3D

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- So any set of either one or two edges is independent, and has rank equal to cardinality.

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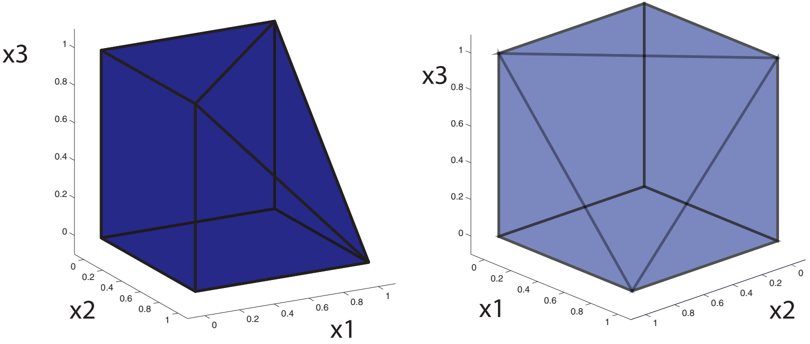
- So any set of either one or two edges is independent, and has rank equal to cardinality.
- The set of three edges is dependent, and has rank 2.

Matroid Polyhedron in 3D

Two view of P_r associated with a matroid $(\{e_1, e_2, e_3\}, \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\})$.



Matroid Polyhedron in 3D

P_r associated with the "free" matroid in 3D.

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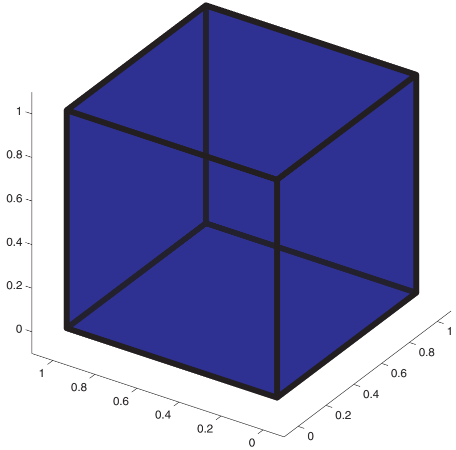

Another Polytope in 3D

Thought question: what kind of polytope might this be?

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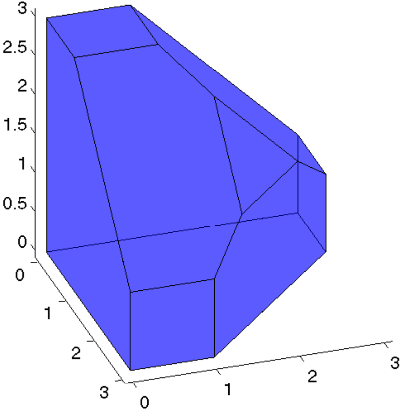

Matroid Independence Polyhedron

- So recall from a moment ago, that we have that

$$P_{\text{ind. set}} = \text{conv}\left\{\cup_{I \in \mathcal{I}} \mathbf{1}_I\right\} \tag{35}$$

$$\subseteq P_r = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{36}$$

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- In fact, the two polyhedra are identical (and thus both are polytopes).
- We'll show this in the next few theorems.

Matroid Independence Polyhedron

Theorem 4.2

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max\{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{37}$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i} \tag{38}$$

Matroid Independence Polyhedron

Proof.

- Firstly, note that for any such $w \in \mathbb{R}^E$, we have

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = (w_1 - w_2) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (w_2 - w_3) \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + (w_{n-1} - w_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \tag{39}$$

...

Matroid Independence Polyhedron

Proof.

- Firstly, note that for any such $w \in \mathbb{R}^E$, we have

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = (w_1 - w_2)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (w_2 - w_3)\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + (w_{n-1} - w_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \quad (39)$$

- If we take w in decreasing order, then each coefficient of the vectors is non-negative (except possibly the last one, w_n).

...

Matroid Independence Polyhedron

Proof.

- Now, again assuming $w \in \mathbb{R}_+^E$, order the elements of V as $(v_1, v_2, \ldots, v_n)$ such that $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$

. . .

Matroid Independence Polyhedron

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- Define the sets U_i based on this order as follows, for $i = 0, \ldots, n$

$$U_i \stackrel{\text{def}}{=} \{v_1, v_2, \ldots, v_i\} \tag{40}$$

Note that

$$\mathbf{1}_{U_0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{1}_{U_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{1}_{U_\ell} = \begin{pmatrix} \left.\begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix}\right\} \ell\times \\ \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} (n-\ell)\times \end{pmatrix}, \text{ etc.}$$

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$$I \stackrel{\text{def}}{=} \{v_i | r(U_i) > r(U_{i-1})\} \tag{41}$$

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- Therefore, I is the output of the greedy algorithm. *since the items v_i are coming in decreasing order, and we only choose the ones that increase the rank, which means they don't violate independence.*

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- And therefore, I is a maximum weight independent set.

...

Matroid Independence Polyhedron

Proof.

- Now, we define λ_i as follows

$$\lambda_i \stackrel{\text{def}}{=} w(v_i) - w(v_{i+1}) \text{ for } i = 1, \dots, n-1 \tag{42}$$

$$\lambda_n \stackrel{\text{def}}{=} w(v_n) \tag{43}$$

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- And the weight of the independent set $w(I)$ is given by

$$w(I) = \sum_{v \in I} w(v) = \tag{45}$$

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- Since we took $v_1, v_2, \ldots$ in decreasing order, for all i, we have $\lambda_i \geq 0$

□

Linear Program LP

Consider the linear programming primal problem

$$\begin{array}{rll} \text{maximize} & w^\intercal x & \\ \text{s.t.} & x_v \geq 0 & (v \in V) \\ & x(U) \leq r(U) & (\forall U \subseteq V) \end{array} \tag{46}$$

Linear Program LP

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And its dual:

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{U \subseteq V} y_U r(U), \\ \text{s.t.} \quad & y_U \geq 0 && (\forall U \subseteq V) \\ & \textstyle\sum_{U \subseteq V} y_U \chi^U \geq w \end{aligned} \tag{47}$$

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Thanks to strong duality, the solutions to these are equal to each other.

Linear Program LP

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- This is identical to the problem

$$\max w^{\intercal} x \text{ such that } x \in P_r \tag{49}$$

where, again, $P_r = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\}$.

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where, again, $P_r = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\}$.

- Therefore, since $P_{\text{ind. set}} \subseteq P_r$, the above problem can only have a larger solution. I.e.,

$$\max w^\intercal x \text{ s.t. } x \in P_{\text{ind. set}} \leq \max w^\intercal x \text{ s.t. } x \in P_r. \tag{50}$$

Polytope equivalence

- Now we have the following relations:

$$\max\left\{w(I) : I \in \mathcal{I}\right\} \leq \max\left\{w^\intercal x : x \in P_{\text{ind. set}}\right\} \tag{51}$$

$$\leq \max\left\{w^\intercal x : x \in P_r\right\} \tag{52}$$

$$\stackrel{\text{def}}{=} \alpha_{\min} = \min\left\{\sum_{U \subseteq V} y_U r(U) : y \geq 0, \sum_{U \subseteq V} y_U \chi^U \geq w\right\} \tag{53}$$

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- Theorem 4.2 states that

$$\max\{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \quad (54)$$

for the chain of U_i's and $\lambda_i \geq 0$ that satisfies $w = \sum_{i=1}^n \lambda_i \mathbf{1}_{U_i}$ (i.e., the r.h.s. of Eq. 54 is feasible w.r.t. the dual LP).

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- Therefore, we also have

$$\max\{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \geq \alpha_{\min} \quad (55)$$

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- And since $w \in \mathbb{R}_+^E$ is an arbitrary direction into the positive orthant, we see that $P_r = P_{\text{ind. set}}$

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- Therefore, all the inequalities above are equalities.
- And since $w \in \mathbb{R}_+^E$ is an arbitrary direction into the positive orthant, we see that $P_r = P_{\text{ind. set}}$
- That is, we have just proved

Theorem 4.3

$$P_r = P_{ind.\ set} \tag{56}$$

Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).

Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- In fact, looking at equations starting at Eq 51, we see that the the LP problem with exponential number of constraints $\max\{w^\intercal x : x \in P_{\text{ind. set}}\}$ is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- In fact, looking at equations starting at Eq 51, we see that the the LP problem with exponential number of constraints $\max \{w^\intercal x : x \in P_{\text{ind. set}}\}$ is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

Theorem 4.4

The LP problem $\max \{w^\intercal x : x \in P_{ind.\ set}\}$ *can be solved exactly using the greedy algorithm.*

- This means that if LP problems have certain structure, they can be solved much easier than immediately implied by the equations.

Base Polytope Equivalence

- Consider convex hull of indicator vectors of bases of a matroid. By the same argument, this will be the same as the following polytope

$$x \geq 0 \tag{57}$$

$$x(A) \leq r(A) \; \forall A \subseteq V \tag{58}$$

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$$x(V) = r(V) \tag{59}$$

- By essentially the same argument as above, we have shown that the convex hull of the incidence vectors of the bases of a matroid is a polytope that can be described by Eq. 57- 59 above.

Scratch Paper

Scratch Paper

Scratch Paper

Sources for Today's Lecture

Korte,Vygen-2005, Vondrak-2010, Schrijver-2003, Oxley-1992, Welsh-1973, Goemans-2010, Vazirani-2001