

EE595A – Submodular functions, their optimization and applications – Spring 2011

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering
Winter Quarter, 2011

<http://ee.washington.edu/class/235/2011wtr/index.html>

Lecture 6 - April 15th, 2011

Announcements

- Reminder, HW1 is on the web page now, at http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/hw1.pdf
- It is due, Tuesday, April 26th, 11:45pm
- All submissions must be done electronically, via our drop box. See the link
<https://catalyst.uw.edu/collectit/dropbox/bilmes/14888>,
or look at the homework on the web page.

Matroid and the greedy algorithm

- Let \mathcal{I} be a set of subsets of E that is down-closed. Consider a non-negative modular weight function $w : E \rightarrow \mathbb{R}_+$, and we want to find the $A \in \mathcal{I}$ that maximizes $w(A)$.
- Greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that $A \cup \{y\} \in \mathcal{I}$ with $w(y)$ as large as possible, stopping when no such y exists.

Theorem 2.1

Let \mathcal{I} be a non-empty collection of subsets of a set E , down-closed (i.e., an independence system). Then the pair (E, \mathcal{I}) is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Matroid and greedy

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- If we stop at a negative value, we'll once again get a maximum weight independent set.

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- We don't need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base.
- If we stop at a negative value, we'll once again get a maximum weight independent set.
- We can instead do **as small as possible** thus giving us a minimum weight independent set/base.

Convex Polyhedra

- Convex polyhedra a large topic, we will only draw what we need.

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Definition 3.1

A subset $P \subseteq \mathbb{R}^E$ is a **polyhedron** if there exists an $m \times n$ matrix A and vector $b \in \mathbb{R}^m$ (for some $m \geq 0$) such that

$$P = \{x : Ax \leq b\} \quad (1)$$

Convex Polyhedra

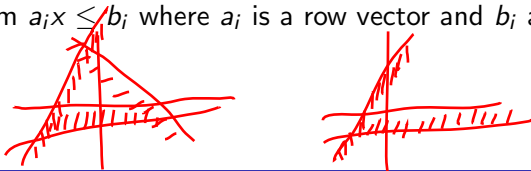
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$$P = \{x : Ax \leq b\} \quad (1)$$

- Thus, P is intersection of finitely many affine halfspaces, which are of the form $a_i x \leq b_i$ where a_i is a row vector and b_i a real scalar.



Convex Polytope

- A polytope is defined as follows

Definition 3.2

A subset $P \subseteq \mathbb{R}^E$ is a **polytope** if it is the convex hull of finitely many vectors in \mathcal{R}^E . That is, if $\exists, x_1, x_2, \dots, x_k \in \mathcal{R}^E$ such that for all $x \in P$, there exists $\{\lambda_i\}$ with $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0 \forall i$ with $x = \sum_i \lambda_i x_i$.

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- We define the convex hull operator as follows:

$$\text{conv}(x_1, x_2, \dots, x_k) \stackrel{\text{def}}{=} \left\{ \underbrace{\sum_{i=1}^k \lambda_i x_i}_{\text{red wavy line}} : \forall i, \lambda_i \geq 0, \text{ and } \sum_i \lambda_i = 1 \right\} \quad (2)$$

Convex Polytope - key representation theorem

- A polytope can be defined in a number of ways, two of which include

Theorem 3.3

A subset $P \subseteq \mathbb{R}^E$ is a polytope iff it can be described in either of the following (equivalent) ways:

- P is the convex hull of a finite set of points.
- If it is a bounded intersection of halfspaces, that is there exists matrix A and vector b such that

$$P = \{x : Ax \leq b\} \quad (3)$$



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$$P = \{x : Ax \leq b\} \tag{3}$$

- This result follows directly from results proven by Fourier, Motzkin, Farkas, and Carathéodory.

Linear Programming

Theorem 3.4 (weak duality)

Let A be a matrix and b and c vectors, then

$$\max \{c^T x \mid Ax \leq b\} \leq \min \{y^T b : y \geq 0, y^T A = c^T\} \quad (4)$$

Linear Programming

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Theorem 3.5 (strong duality)

Let A be a matrix and b and c vectors, then

$$\max \{c^T x \mid Ax \leq b\} \stackrel{!}{=} \min \{y^T b : y \geq 0, y^T A = c^T\} \quad (5)$$

Linear Programming duality forms

$$A = n \times m$$

There are many ways to construct the dual. For example,

$$\max \{c^T x \mid x \geq 0, Ax \leq b\} = \min \{y^T b \mid y \geq 0, y^T A \geq c^T\} \quad (6)$$

Linear Programming duality forms

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Linear Programming duality forms

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Intuitively, why is [one set of equations] the dual of [another quite different set of equations]? In our experience, this is not the right question to be asked. As stated in Section 12.1, there is a purely mechanical procedure for obtaining the dual of a linear program. Once the dual is obtained, one can devise intuitive, and possibly physical meaningful, ways of thinking about it. Using this mechanical procedure, one can obtain the dual of a complex linear program in a fairly straightforward manner. Indeed, the LP-duality-based approach derives its wide applicability from this fact.

Vector, modular, incidence

- Recall, any vector $x \in \mathbb{R}^E$ can be seen as a modular function, as for any $A \subseteq E$, we have

$$x(A) = \sum_{a \in A} x_a \quad (10)$$

Vector, modular, incidence

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- Given an $A \subseteq E$, define the the incidence vector $\mathbf{1}_A \in \{0, 1\}^E$ on the unit hypercube as follows:

$$\mathbf{1}_A \stackrel{\text{def}}{=} \left\{ x \in \{0, 1\}^E : x_i = 1 \text{ iff } i \in A \right\} \quad (11)$$

equivalently,

$$\mathbf{1}_A(j) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } j \in A \\ 0 & \text{if } j \notin A \end{cases} \quad (12)$$

Matroid

Matroid definition once again.

Definition 4.1 (Matroid)

A set system (E, \mathcal{I}) is a **Matroid** if

$$(I1') \quad \emptyset \in \mathcal{I}$$

$$(I2') \quad \forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \text{ (down-closed)}$$

$$(I3') \quad \forall I, J \in \mathcal{I}, \text{ with } |I| > |J|, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I}$$

Independence Polyhedra

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$$P_{\text{ind. set}} = \text{conv} \{ \cup_{I \in \mathcal{I}} \mathbf{1}_I \} \quad (13)$$

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- **N**ow take the rank function r of M , and define the following polyhedron:

$$P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq \underline{E} \right\} \quad (14)$$

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$$P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq V \right\} \quad (14)$$

- First note, take any $x \in P_{\text{ind. set}}$, then we have that $x \in P_r$ (or $P_{\text{ind. set}} \subseteq P_r$).

Matroid

- If $x \in P_{\text{ind. set}}$, then

$$x = \sum_i \lambda_i \mathbf{1}_I \quad (15)$$

for some appropriate vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$.

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- Clearly, for such x , $x \geq 0$.
- Now, for any $A \subseteq E$,

$$x(A) = \underbrace{x^T \mathbf{1}_A} = \sum_i \lambda_i \mathbf{1}_{I_i}^T \mathbf{1}_A \quad (16)$$

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$$x(A) = x^T \mathbf{1}_A = \sum_i \lambda_i \mathbf{1}_I^T \mathbf{1}_A \quad (16)$$

$$\leq \sum_i \lambda_i \max_{j: I_j \subseteq A} \mathbf{1}_j(E) \quad (17)$$

$$\mathbf{1}_{I_j}(E) = \sum_{i \in E} \mathbf{1}_{I_j}(i) = |I_j|$$

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$$= r(A) \quad (19)$$

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$$= r(A) \quad (19)$$

- Thus, $x \in P_r$.

Matroid Polyhedron in 2D

$$P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, \overbrace{x(A) \leq r(A), \forall A \subseteq V} \right\} \quad (20)$$

- Consider this in two dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \quad (21)$$

$$x_1 \leq r(\{v_1\}) \quad (22)$$

$$x_2 \leq r(\{v_2\}) \quad (23)$$

$$x_1 + x_2 \leq r(\{v_1, v_2\}) \quad (24)$$

$$x(\emptyset) \leq r(\emptyset)$$

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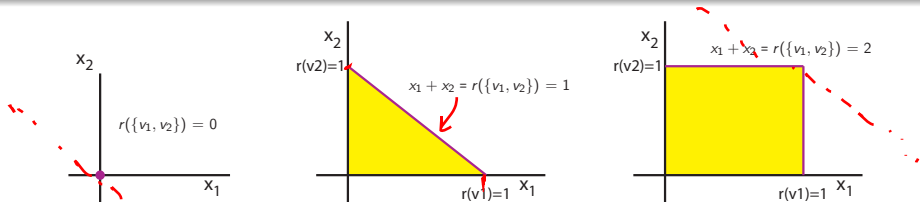
$$x_1 + x_2 \leq r(\{v_1, v_2\}) \quad (24)$$

- Because r is submodular, we have

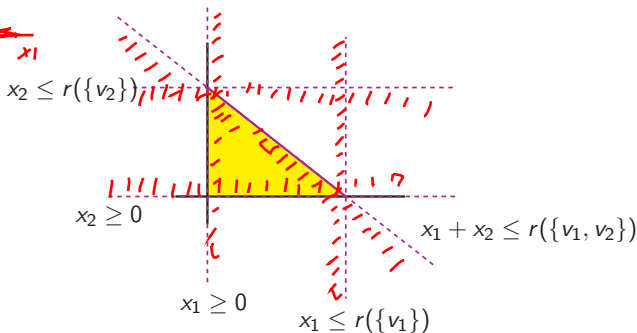
$$r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \quad (25)$$

so since $r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\})$, the last inequality is either touching or active.

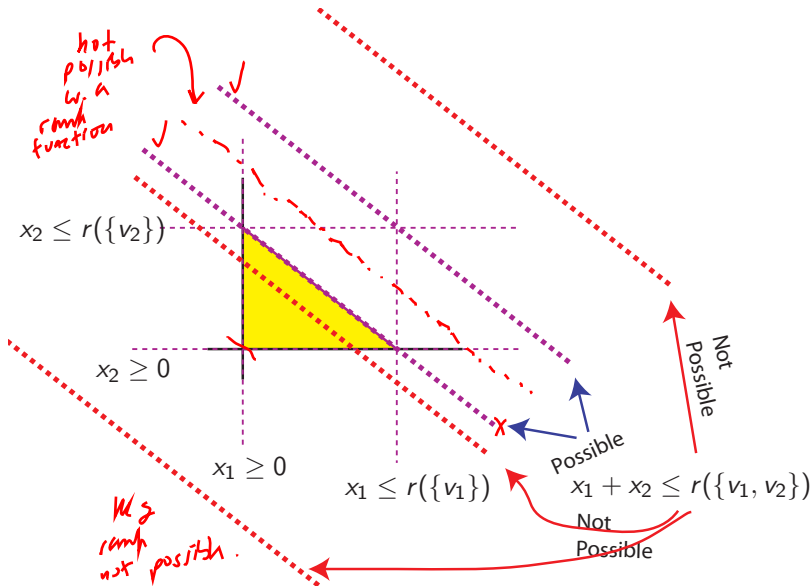
Matroid Polyhedron in 2D



Segment
 v_2 along
 v_1 indep.



Matroid Polyhedron in 2D



Matroid Polyhedron in 3D

$$P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq V \right\} \quad (26)$$

- Consider this in three dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \quad (27)$$

$$x_1 \leq r(\{v_1\}) \quad (28)$$

$$x_2 \leq r(\{v_2\}) \quad (29)$$

$$x_3 \leq r(\{v_3\}) \quad (30)$$

$$x_1 + x_2 \leq r(\{v_1, v_2\}) \quad (31)$$

$$x_2 + x_3 \leq r(\{v_2, v_3\}) \quad (32)$$

$$x_1 + x_3 \leq r(\{v_1, v_3\}) \quad (33)$$

$$x_1 + x_2 + x_3 \leq r(\{v_1, v_2, v_3\}) \quad (34)$$

Matroid Polyhedron in 3D



- Consider the simple cycle matroid on a graph consisting of a 3-cycle, $G = (V, E)$ with matroid $M = (E, \mathcal{I})$ where $I \in \mathcal{I}$ is a forest.

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- So any set of either one or two edges is independent, and has rank equal to cardinality.

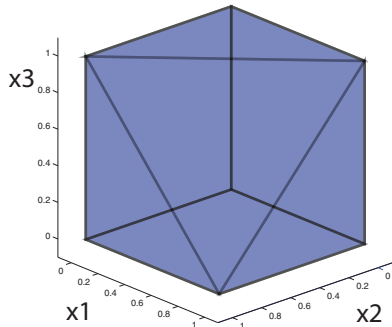
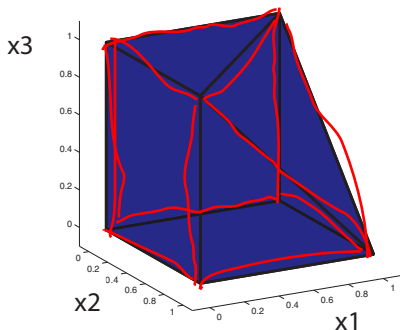
Matroid Polyhedron in 3D

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- So any set of either one or two edges is independent, and has rank equal to cardinality.
- The set of three edges is dependent, and has rank 2.

Matroid Polyhedron in 3D

P_r associated with matroid

$(\{e_1, e_2, e_3\}, \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\})$.

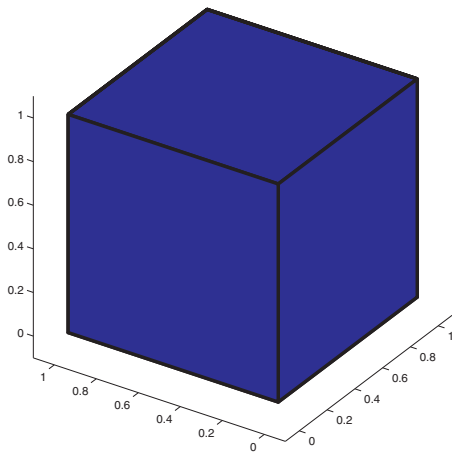


Matroid Polyhedron in 3D

P_r associated with the “free” matroid in 3D.

Matroid Polyhedron in 3D

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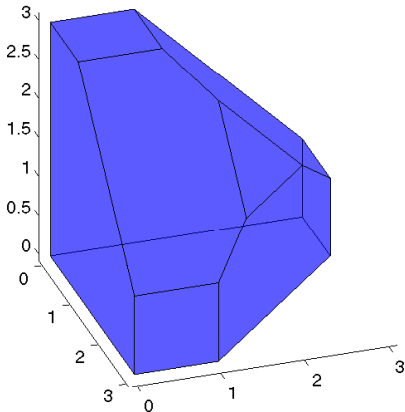


Matroid Polyhedron in 3D

Thought question: what kind of matroid might this be?

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Matroid Independence Polyhedron

- So recall from a moment ago, that we have that

$$P_{\text{ind. set}} = \text{conv} \{ \cup_{I \in \mathcal{I}} \mathbf{1}_I \} \quad (35)$$

$$\subseteq P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq V \right\} \quad (36)$$

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- In fact, the two polyhedra are identical (and thus both are polytopes).
- We'll show this in the next few theorems.

Matroid Independence Polyhedron

Theorem 4.2

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r , then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \dots \subset U_n \subseteq V$ such that

$$\max \{w(I) \mid I \in \mathcal{I}\} = \sum_{i=1}^n \lambda_i r(U_i) \quad (37)$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^n \lambda_i \mathbf{1}_{U_i} \quad (38)$$

Matroid Independence Polyhedron

Proof.

- Firstly, note that for any such $w \in \mathbb{R}^E$, we have

$$\begin{aligned}
 \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} &= (w_1 - w_2) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (w_2 - w_3) \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \\
 &\quad \cdots + (w_{n-1} - w_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \quad (39)
 \end{aligned}$$

...

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$$\begin{aligned}
 \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} &= (w_1 - w_2) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (w_2 - w_3) \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \\
 &\quad \dots + (w_{n-1} - w_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \quad (39)
 \end{aligned}$$

- If we take w in decreasing order, then each coefficient of the vector is non-negative.
($w_i - w_{i+1}$) ≥ 0 except for the last one. ...

Matroid Independence Polyhedron

Proof.

$$w \in \mathbb{R}_+^E$$

- Now, order the elements of V as (v_1, v_2, \dots, v_n) such that $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$

Matroid Independence Polyhedron

Proof.

- Now, order the elements of V as (v_1, v_2, \dots, v_n) such that $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$
- Define the sets U_i based on this order as follows, for $i = 0, \dots, n$

$$U_i \stackrel{\text{def}}{=} \{v_1, v_2, \dots, v_i\} \quad (40)$$

$$I_{U_0} = \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad I_{U_1} = \begin{pmatrix} 1 \\ 0 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad I_{U_e} = \left(\begin{array}{c} 1 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{array} \right) \} e \dots$$

Matroid Independence Polyhedron

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- Define the set I as

$$I \stackrel{\text{def}}{=} \{v_i \mid r(U_i) > r(U_{i-1})\} \quad (41)$$

Matroid Independence Polyhedron

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- Therefore, I is the output of the greedy algorithm. *since the items v_i are coming in decreasing order, and we only choose the ones that increase the rank, which means they don't violate independence.*

Matroid Independence Polyhedron

Proof.

- Now, order the elements of V as (v_1, v_2, \dots, v_n) such that $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$
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- Define the set I as

$$I \stackrel{\text{def}}{=} \{v_i \mid r(U_i) > r(U_{i-1})\} \quad (41)$$

- Therefore, I is the output of the greedy algorithm.
- And therefore, I is a maximum weight independent set.

...

Matroid Independence Polyhedron

Proof.

- Now, we define λ_i as follows

$$\lambda_i \stackrel{\text{def}}{=} w(v_i) - w(v_{i+1}) \text{ for } i = 1, \dots, n-1 \quad (42)$$

$$\lambda_n \stackrel{\text{def}}{=} w(v_n) \quad (43)$$

Matroid Independence Polyhedron

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- And the weight of the independent set $w(I)$ is given by

$$w(I) = \sum_{v \in I} w(v) =$$

(45)

...

Matroid Independence Polyhedron

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$$w(I) = \sum_{v \in I} w(v) = \sum_{i=1}^n w(v_i)(r(U_i) - r(U_{i-1})) \quad (44)$$



(45)

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Matroid Independence Polyhedron

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$$= w(v_n)r(U_n) + \sum_{i=1}^{n-1} (w(v_i) - w(v_{i+1}))r(U_i) \quad (45)$$

$U_0 = \emptyset$

...

Matroid Independence Polyhedron

Proof.

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$$= w(v_n)r(U_n) + \sum_{i=1}^{n-1} (w(v_i) - w(v_{i+1}))r(U_i) = \sum_{i=1}^n \lambda_i r(U_i) \quad (45)$$

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Matroid Independence Polyhedron

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- Since we took v_1, v_2, \dots in decreasing order, for all i , we have $\lambda_i \geq 0$



Linear Program LP

Consider the linear programming primal problem

$$\begin{array}{ll}
 \text{maximize} & w^T x \\
 \text{s.t.} & x_v \geq 0 \quad (v \in V) \\
 & x(U) \leq r(U) \quad (\forall U \subseteq V)
 \end{array} \quad \left. \vphantom{\begin{array}{l} \\ \\ \end{array}} \right] P_r \quad (46)$$

Max $w^T x$,
 s.t. $x \in P_r$
 //

Linear Program LP

Consider the linear programming primal problem

$$\begin{aligned}
 & \text{maximize} && w^T x \\
 & \text{s.t.} && x_v \geq 0 && (v \in V) \\
 & && x(U) \leq r(U) && (\forall U \subseteq V)
 \end{aligned} \tag{46}$$

And its dual:

$$\begin{aligned}
 & \text{minimize} && \sum_{U \subseteq V} y_U r(U), \\
 & \text{s.t.} && y_U \geq 0 && (\forall U \subseteq V) \\
 & && \sum_{U \subseteq V} y_U x^U \geq w
 \end{aligned} \tag{47}$$

Linear Program LP

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 & && \sum_{U \subseteq V} y_U \chi^U \geq w
 \end{aligned} \tag{47}$$

Thanks to strong duality, the solutions to these are equal to each other.

Linear Program LP

- Consider the linear programming primal problem

$$\begin{aligned} & \text{maximize} && w^T x \\ & \text{s.t.} && x_v \geq 0 && (v \in V) \\ & && x(U) \leq r(U) && (\forall U \subseteq V) \end{aligned} \tag{48}$$

Linear Program LP

- Consider the linear programming primal problem

$$\begin{aligned}
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 \end{aligned} \tag{48}$$

- This is identical to the problem

$$\max w^T x \text{ such that } x \in P_r \tag{49}$$

where, again, $P_r = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq V\}$. ✓

Linear Program LP

- Consider the linear programming primal problem

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- This is identical to the problem

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where, again, $P_r = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq V\}$.

- Therefore, since $P_{\text{ind. set}} \subseteq P_r$, the above problem can only have a larger solution.

$$\begin{aligned}
 & \text{ind.} \\
 & \max w^T x \\
 & \text{s.t. } x \in P_{\text{ind. set}}
 \end{aligned}
 \leq
 \begin{aligned}
 & \max w^T x \\
 & \text{s.t. } x \in P_r
 \end{aligned}$$

Polytope equivalence

- Now we have the following relations:

$$\max \{w(I) : I \in \mathcal{I}\} \leq \max \{w^T x : x \in P_{\text{ind. set}}\} \quad (50)$$

$$\leq \max \{w^T x : x \in P_r\} \quad (51)$$

$$\stackrel{\text{def}}{=} \alpha_{\min} = \min \left\{ \sum_{U \subseteq V} y_U r(U) : y \geq 0, \sum_{U \subseteq V} y_U x^U \geq w \right\} \quad (52)$$

Polytope equivalence

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- Theorem 4.2 states that

$$\max \{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^n \lambda_i r(U_i) \quad (53)$$

for the chain of U_i 's and $\lambda_i \geq 0$ that satisfies $w = \sum_{i=1}^n \lambda_i \mathbf{1}_{U_i}$

i.e., (53) is feasible w.r.t. the dual LP.

Polytope equivalence

- Now we have the following relations:

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for the chain of U_i 's and $\lambda_i \geq 0$ that satisfies $w = \sum_{i=1}^n \lambda_i \mathbf{1}_{U_i}$

- But we have

$$\max \{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^n \lambda_i r(U_i) \geq \alpha_{\min} \quad (54)$$

Polytope equivalence

- Now we have the following relations:

$$\max \{w(I) : I \in \mathcal{I}\} \leq \max \{w^T x : x \in P_{\text{ind. set}}\} \quad (50)$$

$$\leq \max \{w^T x : x \in P_r\} \quad (51)$$

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- Therefore, all the inequalities above are equalities.

Polytope equivalence

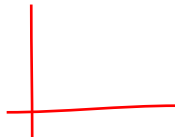
- Now we have the following relations:

$$\max \{w(I) : I \in \mathcal{I}\} = \max \{w^T x : x \in P_{\text{ind. set}}\} \quad (50)$$

$$= \max \{w^T x : x \in P_r\} \quad (51)$$

$$\stackrel{\text{def}}{=} \alpha_{\min} = \min \left\{ \sum_{U \subseteq V} y_U r(U) : y \geq 0, \sum_{U \subseteq V} y_U x^U \geq w \right\} \quad (52)$$

- Therefore, all the inequalities above are equalities.
- And since $w \in \mathbb{R}_+^E$ is an arbitrary direction into the positive orthant, we see that $P_r = P_{\text{ind. set}}$



Polytope equivalence

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).

Polytope equivalence

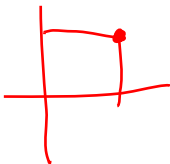
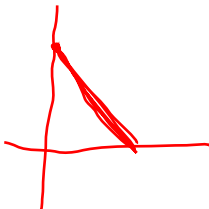
- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- Consider convex hull of indicator vectors of bases of a matroid. By the same argument, this will be the same as the following polytope

$$x \geq 0 \quad (55)$$

$$x(A) \leq r(A) \quad \forall A \subseteq V \quad (56)$$

$$x(V) = r(V) \quad (57)$$

$\left. \begin{array}{l} (55) \\ (56) \end{array} \right\} B_r$



Scratch Paper

Scratch Paper

Scratch Paper

Sources for Today's Lecture

Korte, Vygen-2005, Vondrak-2010, Schrijver-2003, Oxley-1992,
Welsh-1973, Goemans-2010, Vazirani-2001