EE595A – Submodular functions, their optimization and applications – Spring 2011

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http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 9 - April 29th, 2011

Announcements

- HW2 should be hopefully ready by this weekend (I'll send email when ready).
- On Final projects. **One** single page final project proposals (revision one) are due next Friday (one week from today) at 6:00pm.
- Again, all submissions must be done electronically, via our drop box. See the link `https://catalyst.uw.edu/collectit/dropbox/bilmes/14888`, or look at the homework on the web page.
- Email me and/or stop by office hours for ideas. The proposals next Friday are non-binding (you can change your mind later) but you should start thinking about project proposals now.
- Ideal proposal would lead to a NIPS paper in June and be related to submodularity.

A polymatroid function's polyhedron is a polymatroid.

Theorem 2.1

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f-basis y^x of x, we have that the component sum of y is

$$\max\left(y(E) : y \leq x, y \in P_f\right) = y^x(E) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{1}$$

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There, as we will see, are a number of consequences of this theorem (other than that P_f is a polymatroid).

Matroid case

- Considering the above theorem, the matroid case is now a special case, where we have that:

Corollary 2.2

We have that:

$$\max \{y(E) : y \in P_{ind.\ set}(M), y \leq x\} = \min \{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{2}$$

where r_M is the matroid rank function of some matroid.

Matroid from submodular function

Theorem 2.3

Given integral polymatroid function f, let $(E, \mathcal{F})$ be a set system with ground set E and set of subsets $\mathcal{F}$ such that

$$\forall J \in F, \forall \emptyset \subset S \subseteq J, |S| \leq f(S) \tag{3}$$

Then $M = (E, \mathcal{F})$ is a matroid.

Proof.

Exercise □

And its rank function is Exercise.

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- This corresponds to $\min \{r_M(A) + x(E \setminus A) : A \subseteq E\}$ since x is modular and $x(E \setminus A) = x(E) - x(A)$.
- More importantly, $\min \{r_M(A) + x(E \setminus A) : A \subseteq E\}$ a form of submodular function minimization, namely $\min \{r_M(A) - x(A) : A \subseteq E\}$ for a submodular function consisting of a difference of matroid rank and modular (so no longer nec. monotone, nor positive).

Problem To Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathcal{R}_+^E$;

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- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*

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- By the above theorem, the existence of such an A will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, A is minimal in terms of $f(A) \stackrel{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).

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- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of f at A.
- This will also run in polynomial time.

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- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.
- It's going to take us a few lectures to fully develop this algorithm, so please keep mind of the overall goal.

Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.

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- Next choose a $v_3 \in \text{span}_1(I) - \text{span}_1(I - v_2)$ to recover what was lost in $I \cup \{v_1\}$ when we removed v_2 from it.

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- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \text{span}_1(I)$, so $\text{span}_1(I + v_1) = \text{span}_1(I + v_1 - v_2 + v_3)$.

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- But $I + v_1 - v_2 + v_3$ might not be independent in M_2 again, so we need to find an $v_4 \in C_2(I + v_1 - v_2, v_3)$ to remove, and so on.

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- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \ldots, v_n)$ such that we will be independent in both M_1 andM_2 and thus be one greater in size than I.

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- We then replace I with $I \ominus S$ (quite analogous to the bipartite matching case), and start again.

Alternating and Augmenting Sequences

- Let I be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

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- Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for i odd, and $e_i \in I$ for i even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that S is an alternating sequence w.r.t. I if the following are true.

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    3. For all odd i, $\text{span}_1(I \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.

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  3. For all odd i, $\text{span}_1(I \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.
- Lastly, if also, $|S| = s$ is odd, and $I \ominus S \in \mathcal{I}_2$, then S is called an augmenting sequence w.r.t. I.

Alternating and Augmenting Sequences

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- Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover, if there is an augmenting sequence, then the intersection is not maximum.
- We next wish to show that, if the intersection is maximum, then there is an augmenting sequence.

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- The algorithm described becomes:

Algorithm 9.1: Alternating Path Matroid Intersection

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Let I be an arbitrary (including empty) independent set in two matroids M_1 and M_2 ;
while There exists an augmenting sequence S do
    I ← I ⊖ S ;
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Matroid Intersection Algorithm

- Given two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$
- Our goal is to find the maximum cardinality set I such that $I \in \mathcal{I}_1$ and $I \in \mathcal{I}_2$.
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Algorithm 9.1: Alternating Path Matroid Intersection

1 Let I be an arbitrary (including empty) independent set in two matroids M_1 and M_2 ;
2 **while** *There exists an augmenting sequence* S **do**
3 $\quad I \leftarrow I \ominus S$;

- This can be made to run in $O(m^2 R + mR^2 c(m))$ where $m = |V|$, $R =$ max rank of the two matroids, and $c(m)$ is the max independence testing cost for the two matroids, but faster algorithms exist as well (see Schrijver-2003).

Border graphs

- We construct an auxiliary directed bipartite graph (border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current I, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to I, or from I back to $E \setminus I$.

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- Left-going edges: For each $e_i \in \text{span}_1(I) \setminus I$, create $\leftarrow$ edge $(e_j, e_i) \in Z$ for any $e_j \in C_1(I, e_i) \setminus \{e_i\}$.

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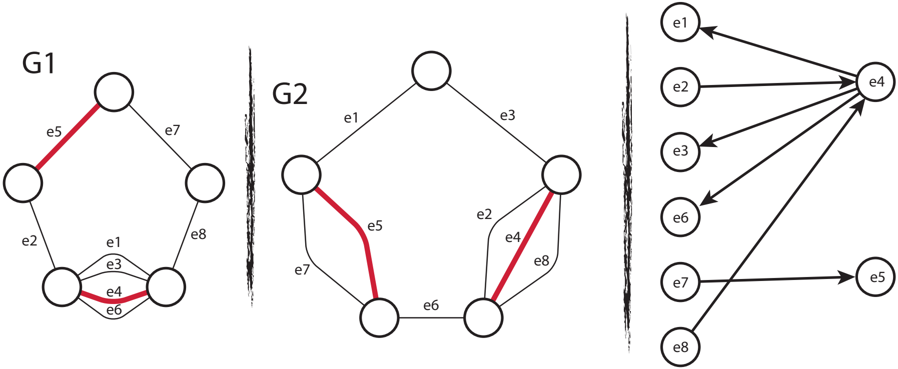
Border graph Example



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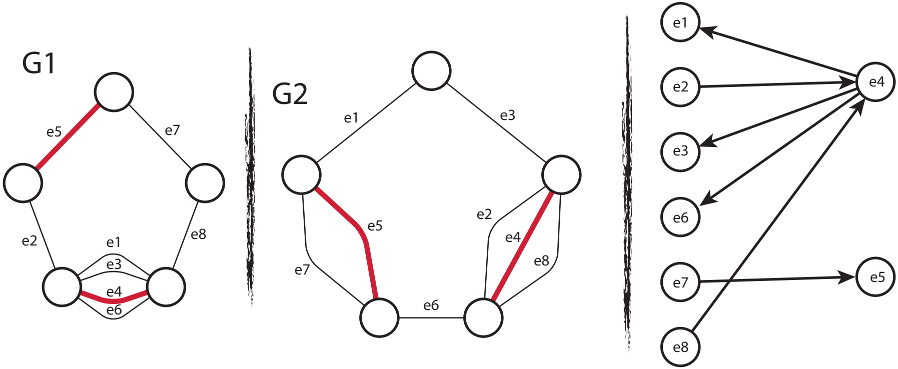

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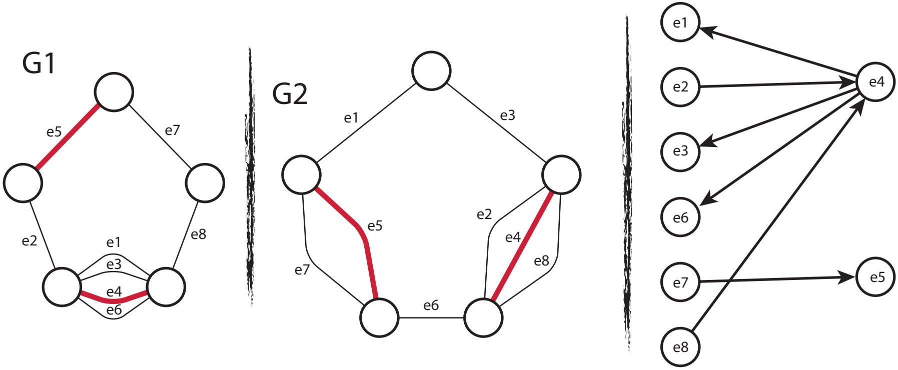

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- Are there others?

Identifying Augmenting Sequences

Lemma 3.1

If S is a source-sink path in $B(I)$, and there is no shorter source-sink path between the same source and sink (i.e., there are no short-cuts), then S is an augmenting sequence w.r.t. I.

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Lemma 3.2

Let I and J be intersections such that $|I| + 1 = |J|$. Then there exists a source-sink path S in $B(I)$ where $S \subseteq I \ominus J$.

Identifying Augmenting Sequences

Theorem 3.3

Let I_p and I_{p+1} be intersections of M_1 and M_2 with p and $p+1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \ominus I_{p+1}$ w.r.t. I_p.

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Theorem 3.5

For any intersection I, there exists a maximum cardinality intersection I^* such that $\text{span}_1(I) \subseteq \text{span}_1(I^*)$ and $\text{span}_2(I) \subseteq \text{span}_2(I^*)$.

Matroid Partition Problem

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- We wish to, if possible, partition E into k blocks, $I_i, i \in \{1, 2, \dots, k\}$ where $I_i \in \mathcal{I}_i$.
- Moreover, we want partition to be lexicographically maximum, that is $|I_1|$ is maximum, $|I_2|$ is maximum given $|I_1|$, and so on.

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$$|A| \leq kr(A) \;\; \forall A \subseteq E \tag{6}$$
- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as
$$\frac{1}{k}\mathbf{1}(A) \leq r(A) \;\; \forall A \subseteq E \tag{7}$$

Matroid Partition Problem and Submodular Function Minimization

- Recall definition of matroid polytope

$$P_r = \left\{ y \in \mathbb{R}_+^E : y(A) \leq r(A) \text{ for all } A \subseteq E \right\} \tag{11}$$

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- We also see that this is essentially a special case of submodular function minimization, namely finding A that minimizes $r(A) - \frac{1}{k}\mathbf{1}(A)$.
- In the general case, we are looking for an A that minimizes $\sum_i r_i(A) - \mathbf{1}(A)$, and a sum of submodular functions is submodular (in fact, a sum of matroid rank functions is a type of polymatroid rank function Exercise).

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- and moreover for each independent set I_i, we have

$$|I_i \cap A| \leq r_i(A) \tag{9}$$

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- which immediately gives:

$$|A| \leq \sum_{i=1}^{k} r_i(A) \quad (10)$$

  proving the first half of the theorem.

□

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- We give algorithm for $I = E$, but any $I \subseteq E$ can be used instead.

Matroid Partition Algorithm (and proof)

1. The algorithm starts with a set of k empty sets, $I_k = \emptyset, i = 1, 2, \ldots, k$. J is our index set, so $J = \{1, 2, \ldots, k\}$.

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2. We are going to create a sequence of subsets $(S_0, S_1, \ldots)$, starting with $S_0 = E$, and the others are defined by the algorithm below. *For each S_i, there will be an associated $I_{j(i)}$, giving mapping $j : \mathbb{Z}_+ \rightarrow J$. For simplicity, we will be calling/renaming as follows: $I_{j(i)}$ will be renamed/referred-to-by just I_i w.l.o.g. We note that there might be duplicates, meaning that we could (and probably will) have that $j(i) = j(i')$ for $i \neq i'$ which means that, after our renaming, we'll likely have that $I_i = I_{i'}$ for certain pairs (i, i') with $i \neq i'$.*

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7. Therefore, $\exists$ smallest i' s.t. $|I_{i'} \cap S_1| < r_{i'}(S_1)$. W.l.o.g. name $i' = 2$, and let $S_2 \stackrel{\text{def}}{=} S_1 \cap \text{span}_2(I_2 \cap S_1)$.

Matroid Partition Algorithm (and proof)

8. Note $S_2 \subseteq S_1$. Moreover, we have $S_2 \subset S_1$ (proper) since

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10. Therefore, $\exists$ smallest i' s.t. $|I_{i'} \cap S_j| < r_i(S_j)$. W.l.o.g. name $i' = j + 1$, and let $S_{j+1} \stackrel{\text{def}}{=} S_j \cap \text{span}_{j+1}(I_{j+1} \cap S_j)$.

Matroid Partition Algorithm (and proof)

8. Note $S_2 \subseteq S_1$. Moreover, we have $S_2 \subset S_1$ (proper) since
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r_2(S_2) &= r_2(S_1 \cap \text{span}_2(I_2 \cap S_1)) && (12)\\
&\leq r_2(\text{span}_2(I_2 \cap S_1)) && (13)\\
&= r_2(I_2 \cap S_1) && (14)\\
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12. In general, we have $S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_{h-1}$ as long as $e \in S_j$ for $j = 0, \ldots, h-1$. The I_i's might be reused (so could have $h > k$).

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17. Thus, $\exists$ and we chose an element $e' \in C_h(I_h, e) \setminus S_{h-1}$. Note that $e' \neq e$ because $e \in S_j$ for all $j < h$.

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18. Then, we update $I_h \leftarrow I_h + e - e'$ which retains I_h's independence.
19. Next, decrement $h \leftarrow h - 1$, update $e \leftarrow e'$, and GOTO line 14.

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22. Therefore, decrement will terminate at some point at or before we hit $h = 1$. *Exercise: argue that if we reach some maximally independent set in some $h > 1$, then the algorithm will terminate before we reach $h = 1$.*

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22. Therefore, decrement will terminate at some point at or before we hit $h = 1$.
23. This completes the algorithm, and the algorithmic proof.

Matroid Partition Problem

Theorem 4.1

Let M_i be a collection of k matroids as described. Then, a set $I \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \dots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A) \tag{5}$$

where r_i is the rank function of M_i.

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- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as
$$\frac{1}{k}\mathbf{1}(A) \leq r(A) \;\; \forall A \subseteq E \tag{7}$$

Matroid Partition - Flow solution when $M = M_i, \forall i$

- It extends partition $(I_i : i \in J)$ of a proper subset of E into k independent sets, to such a partitioning of a larger subset.

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- Create directed edge (e, t) for all $e \in E$ such that $\exists i \in J$ with $e \notin I_i$ **and** $I_i + e \in \mathcal{I}$. I.e., we add this edge (e, t) if there is some independent set I_i that remains independent if e is added to it.

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- Add directed edge (e, f) for any distinct $e, f \in E$ such that $I_i + e \notin \mathcal{I}$ and $f \in C(I_i, e)$ for some i. That is, we add an edge (e, f) where e directs **to** the elements of a (nec. unique) circuit that is potentially created when e is added to I_i for some i.

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- If the shortest path is $S = (s, e, t)$ then we can add e to some independent set and it is still independent.
- If the shortest path is $S = (s, e, f, t)$ then we can add e to some I_1, create a circuit, but that gets broken when we remove f from that circuit rendering I_1 once again independent, but then there must be some other I_2 that f can be added to w/o making I_2 independent. Thus, the new independent sets are $I_1 + e - f$ and $I_2 + f$, thus we are making progress.

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  3. add f_1 to some I_2, thus making a circuit C_2 due to edge (f_1, f_2).

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  3. add f_1 to some I_2, thus making a circuit C_2 due to edge (f_1, f_2).
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  5. add f_2 to some I_3, not making a circuit due to edge (f_2, t).

  thus making progress.

Flow solution theorem

Thus, we have outlined the proof of one direction in the following theorem. When all matroids are the same $\forall i, M_i = M$ for some matroid, we have:

Theorem 4.2

There is an (s, t) path in the aforementioned graph iff the set of independent sets $(I_i : i \in J)$ can be grown by one element and still be a partition of some subset of E.

The other direction can be shown as a consequence of Theorem 4.1. Exercise

Problem To Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathcal{R}_+^E$;
- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*
- By the above theorem, the existence of such an A will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, A is minimal in terms of $f(A) \stackrel{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).
- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of f at A.
- This will also run in polynomial time.

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- We gradually build up y by adding new independent sets (and augmenting J), adding to the existing independent sets, and adjusting coefficients.
- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we'll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.

Associated digraph for polyhedra membership

- Define associated digraph G as follows.

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- Add directed edge (e, f) for any distinct $e, f \in E$ such that $I_i + e \notin \mathcal{I}$ and $f \in C(I_i, e)$ for some i. That is, we add an edge (e, f) where e directs **to** the elements of a (nec. unique) circuit that is potentially created when e is added to I_i for some i.

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- The algorithm starts with $y = 0$, $J = \{0\}$, $I_0 = \emptyset$, and $\lambda_0 = 1$.

Augmenting path theorem

Theorem 5.1

If there is a directed path from s to t in G, then there exists $y' \in P$ with $y \le y' \le x$, with $y'(E) \ge y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

We will prove this next time.

Augmenting path theorem consequences

Corollary 5.2

For any $x \in \mathbb{R}_+^E$, we have

$$\max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (16)$$

Note: this was not used in the theorem above, rather it is a consequence!

Proof.

- First, any $y \in P$ with $y \leq x$, and any $A \subset E$, we have

$$y(E) = y(A) + y(E \setminus A) \leq r(A) + x(E \setminus A) \quad (17)$$

as we have seen.

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- Then there exists no such $y' \in P$ s.t. $y'(E) > y(E)$, and the digraph won't have a directed path from s to t (by the theorem).

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- So we need only find a y giving equality.
- Choose any $y \in P$ such that $y \le x$ and with $y(E)$ maximum.
- Then there exists no such $y' \in P$ s.t. $y'(E) > y(E)$, and the digraph won't have a directed path from s to t (by the theorem).
- Then, there is a set A such that $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$, or that $y(E) = r(A) + x(E \setminus A)$, thus demonstrating equality.

□

Scratch Paper

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Sources for Today's Lecture

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