

EE595A – Submodular functions, their optimization and applications – Spring 2011

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http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 9 - April 29th, 2011

Announcements

- HW2 should be hopefully ready by this weekend (I'll send email when ready).
- On Final projects. **One** single page final project proposals (revision one) are due next Friday (one week from today) at 6:00pm.
- Again, all submissions must be done electronically, via our drop box. See the link
<https://catalyst.uw.edu/collectit/dropbox/bilmes/14888>, or look at the homework on the web page.
- Email me and/or stop by office hours for ideas. The proposals next Friday are non-binding (you can change your mind later) but you should start thinking about project proposals now.
- Ideal proposal would lead to a NIPS paper in June and be related to submodularity.

A polymatroid function's polyhedron is a polymatroid.

Theorem 2.1

Let f be a polymatroid function defined on subsets of E . For any $x \in \mathbb{R}_+^E$, and any P_f -basis y^x of x , we have that the component sum of y is

$$\max(y(E) : y \leq x, y \in P_f) = y^x(E) = \min(x(A) + f(E \setminus A) : A \subseteq E) \quad (1)$$

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There, as we will see, are a number of consequences of this theorem (other than that P_f is a polymatroid).

Matroid case

- Considering the above theorem, the matroid case is now a special case, where we have that:

Corollary 2.2

We have that:

$$\max \{y(E) : y \in P_{ind. set}(M), y \leq x\} = \min \{r_M(A) + x(E \setminus A) : A \subseteq E\} \quad (2)$$

where r_M is the matroid rank function of some matroid.

Matroid from submodular function

Theorem 2.3

Given integral polymatroid function f , let (E, \mathcal{F}) be a set system with ground set E and set of subsets \mathcal{F} such that

$$\forall J \in \mathcal{F}, \forall \emptyset \subset S \subseteq J, |S| \leq f(S) \quad (3)$$

Then $M = (E, \mathcal{F})$ is a matroid.

Proof.

Exercise □

And its rank function is **Exercise**.

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- This corresponds to $\min \{r_M(A) + x(E \setminus A) : A \subseteq E\}$ since x is modular and $x(E \setminus A) = x(E) - x(A)$.
- More importantly, $\min \{r_M(A) + x(E \setminus A) : A \subseteq E\}$ a form of submodular function minimization, namely $\min \{r_M(A) - x(A) : A \subseteq E\}$ for a submodular function consisting of a difference of matroid rank and modular (so no longer nec. monotone, nor positive).

Problem To Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathcal{R}_+^E$;

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- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M , and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*



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- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of f .
- This will also run in polynomial time.

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- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.
- It's going to take us a few lectures to fully develop this algorithm, so please keep mind of the overall goal.

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- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \text{span}_1(I)$, so $\text{span}_1(I + v_1) = \text{span}_1(I + v_1 - v_2 + v_3)$.

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- But $I + v_1 - v_2 + v_3$ might not be independent in M_2 again, so we need to find an $v_4 \in C_2(I + v_1 - v_2, v_3)$ to remove, and so on.

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- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \dots, v_n)$ such that we will be independent in both M_1 and M_2 and thus be one greater in size than I .

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- We then replace I with $I \oplus S$ (quite analogous to the bipartite matching case), and start again.

Alternating and Augmenting Sequences

- Let I be an **intersection** of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

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- Lastly, if also, $|S| = s$ is odd, and $I \ominus S \in \mathcal{I}_2$, then S is called an **augmenting sequence** w.r.t. I .

Alternating and Augmenting Sequences

- If I admits an augmenting sequence S , then the above argument shows that $I \ominus S$ is independent in M_1 , independent in M_2 , and also we have that $|I| + 1 = |I \ominus S|$.

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- Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover, if there is an augmenting sequence, then the intersection is not maximum.
- We next wish to show that, if the intersection is maximum, then there is an augmenting sequence.

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- Our goal is to find the maximum cardinality set I such that $I \in \mathcal{I}_1$ and $I \in \mathcal{I}_2$.
- The algorithm described becomes:

Algorithm 9.1: Alternating Path Matroid Intersection

- 1 Let I be an arbitrary (including empty) independent set in two matroids M_1 and M_2 ;
 - 2 **while** *There exists an augmenting sequence S* **do**
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Matroid Intersection Algorithm

- Given two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$
- Our goal is to find the maximum cardinality set I such that $I \in \mathcal{I}_1$ and $I \in \mathcal{I}_2$.
- The algorithm described becomes:

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- This can be made to run in $O(m^2R + mR^2c(m))$ where $m = |V|$, $R = \max$ rank of the two matroids, and $c(m)$ is the max independence testing cost for the two matroids, but faster algorithms exist as well (see Schrijver-2003).

Border graphs

- We construct an auxiliary directed bipartite graph (border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current I , that will help us with this problem. The graph has only directed edges from $E \setminus I$ to I , or from I back to $E \setminus I$.

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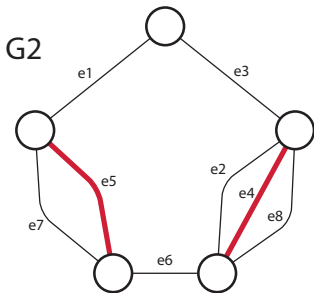
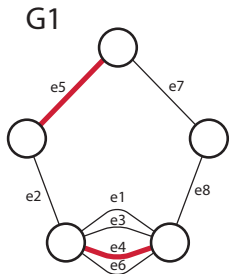
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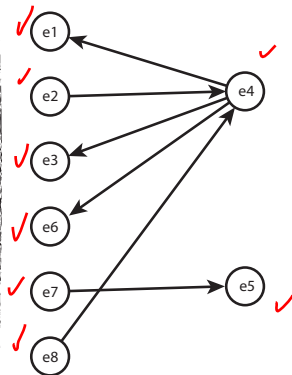
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Border graph Example

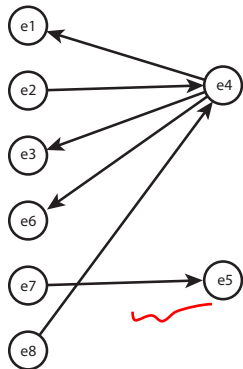
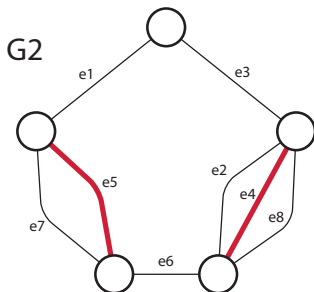
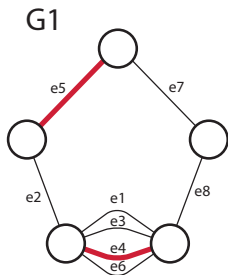


CS



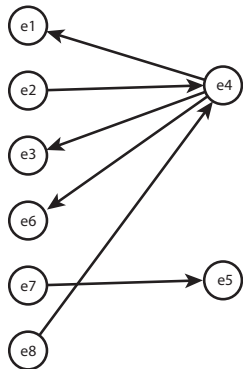
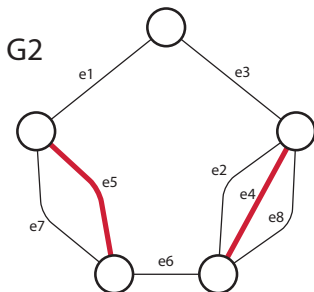
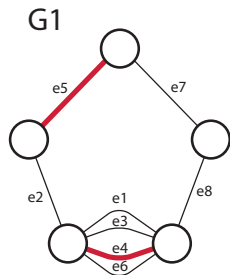
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- Are there others? *Yes.*

Identifying Augmenting Sequences

Lemma 3.1

If S is a source-sink path in $B(I)$, and there is no shorter source-sink path between the same source and sink (i.e., there are no short-cuts), then S is an augmenting sequence w.r.t. I .

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Lemma 3.2

Let I and J be intersections such that $|I| + 1 = |J|$. Then there exists a source-sink path S in $B(I)$ where $S \subseteq I \oplus J$.

Identifying Augmenting Sequences

Theorem 3.3

Let I_p and I_{p+1} be intersections of M_1 and M_2 with p and $p + 1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \ominus I_{p+1}$ w.r.t. I_p .

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Theorem 3.5

For any intersection I , there exists a maximum cardinality intersection I^ such that $\text{span}_1(I) \subseteq \text{span}_1(I^*)$ and $\text{span}_2(I) \subseteq \text{span}_2(I^*)$.*

Matroid Partition Problem

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- We wish to, if possible, partition E into k blocks, $I_i, i \in \{1, 2, \dots, k\}$ where $I_i \in \mathcal{I}_i$.
- Moreover, we want partition to be lexicographically maximum, that is $|I_1|$ is maximum, $|I_2|$ is maximum given $|I_1|$, and so on.

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Let M_i be a collection of k matroids as described. Then, a set $I \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \dots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i , if and only if, for all $A \subseteq I$

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where r_i is the rank function of M_i .

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- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as

$$\frac{1}{k} \mathbf{1}(A) \leq r(A) \quad \forall A \subseteq E \quad (7)$$

Matroid Partition Problem and Submodular Function Minimization

- Recall definition of matroid polytope

$$P_r = \left\{ y \in \mathbb{R}_+^E : y(A) \leq r(A) \text{ for all } A \subseteq E \right\} \quad (11)$$

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- We also see that this is essentially a special case of submodular function minimization, namely finding A that minimizes $r(A) - \frac{1}{k}\mathbf{1}(A)$.
- In the general case, we are looking for an A that minimizes $\sum_i r_i(A) - \mathbf{1}(A)$, and a sum of submodular functions is submodular (in fact, a sum of matroid rank functions is a type of polymatroid rank function **Exercise**).

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- and moreover for each independent set I_i , we have

$$|I_i \cap A| \leq r_i(A) \quad (9)$$

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- which immediately gives:

$$|A| \leq \sum_{i=1}^k r_i(A) \quad (10)$$

proving the first half of the theorem.



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- We give algorithm for $I = E$, but any $I \subseteq E$ can be used instead.

Matroid Partition Algorithm (and proof)

- 1 The algorithm starts with a set of k empty sets,
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For each S_i , there will be an associated

$I_{j(i)}$. For simplicity, we will be calling / renaming

$I_{j(i)} \stackrel{M}{=} I_i$ w.l.o.g. could have

there might be duplicates, i.e., $\sqrt{j(i) = j(i')}$
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$$\left(\{e\} + (S_1 \cap \cup_i I_i) \right) \Rightarrow |S_1| > \sum_i r_i(S_1)$$

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- ⑦ Therefore, \exists smallest i' s.t. $|I_{i'} \cap S_1| < r_{i'}(S_1)$. W.l.o.g. name $i' = 2$, and let $S_2 \stackrel{\text{def}}{=} S_1 \cap \text{span}_2(I_2 \cap S_1)$.

Matroid Partition Algorithm (and proof)

8 Note $S_2 \subseteq S_1$. Moreover, we have $S_2 \subset S_1$ (proper) since

$$r_2(S_2) = r_2(S_1 \cap \text{span}_2(I_2 \cap S_1)) \quad \text{by def.} \quad (12)$$

$$\leq r_2(\text{span}_2(I_2 \cap S_1)) \quad \text{mon. of rank} \quad (13)$$

$$= r_2(I_2 \cap S_1) \quad \text{span is r. l.u.s.} \quad (14)$$

$$\stackrel{\text{rank bound}}{\leq} |I_2 \cap S_1| < r_2(S_1) \quad \text{by assumption} \quad (15)$$

and if the rank decreases, the sets can't be equal. ★

$$\therefore S_2 \subset S_1$$

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$$= r_2(I_2 \cap S_1) \quad (14)$$

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and if the rank decreases, the sets can't be equal.

- 9 Iterating on j , assume $e \in S_j$. Now, similarly, if $\forall i, |I_i \cap S_j| \geq r_i(S_j)$, then $|S_j| \geq |\{e\} + \bigcup_i (I_i \cap S_j)| > \sum_i r_i(S_j)$ which is again not possible by assumption (recall $\forall i, e \notin I_i$). $\Rightarrow |S_j| > \sum_i r_i(S_j)$

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- 10 Therefore, \exists smallest i' s.t. $|I_{i'} \cap S_j| < r_{i'}(S_j)$. W.l.o.g. name $i' = j + 1$, and let $S_{j+1} \stackrel{\text{def}}{=} S_j \cap \text{span}_{j+1}(I_{j+1} \cap S_j)$.

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- 11 And we have $S_{j+1} \subset S_j$. *by same argument as equations 12-15.*

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- 12 In general, we have $S_0 \supset S_1 \supset S_2 \supset \dots \supset S_{h-1}$ as long as $e \in S_j$ for $j = 0, \dots, h-1$. Note that the I_i 's might be being reused.

so might have $h > R$

Matroid Partition Algorithm (and proof)

- 13 Due to the strict monotone decreasing, at some point we must get to an h such that $e \notin S_h$. Two things can then happen:

(could be that $S_h = \emptyset$).

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- 14 If $I_h + e \in \mathcal{I}_h$ then set $I_h \leftarrow I_h + e$, thereby growing our set of independent sets. Empty all S_i 's, and GOTO line 3, and continue.

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- 17 Thus, we chose an element $e' \in C_h(I_h, e) \setminus S_{h-1}$. Note that $e' \neq e$ because $e' \in S_j$ for all $j < h$.

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- 18 Then, we update $I_h \leftarrow I_h + e - e'$ which retains I_h 's independence.
- 19 Next, decrement $h \leftarrow h - 1$, update $e \leftarrow e'$, and GOTO line 14.

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- 22 Therefore, decrement will terminate at some point at or before we hit $h = 1$.

if we reach some max indep set
 in some $h > 1$, it will
 terminate before we reach $h = 1$.

Exercise:

Matroid Partition Algorithm (and proof)

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- 22 Therefore, decrement will terminate at some point at or before we hit $h = 1$.
- 23 This completes the algorithm, and the algorithmic proof.

Matroid Partition Problem

Theorem 4.1

Let M_i be a collection of k matroids as described. Then, a set $I \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \dots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i , if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^k r_i(A) \quad (5)$$

where r_i is the rank function of M_i .

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- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as

$$\frac{1}{k} \mathbf{1}(A) \leq r(A) \quad \forall A \subseteq E \quad (7)$$

Matroid Partition - Flow solution when $M = M_i, \forall i$

- It extends partition $(I_i : i \in J)$ of a proper subset of E into k independent sets, to such a partitioning of a larger subset.

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- Create directed edge (e, t) for all $e \in E$ such that $\exists i \in J$ with $e \notin I_i$ **and** $I_i + e \in \mathcal{I}$. I.e., we add this edge (e, t) if there is some independent set I_i that remains independent if e is added to it.

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- Create directed edge (e, t) for all $e \in E$ such that $\exists i \in J$ with $e \notin I_i$ **and** $I_i + e \in \mathcal{I}$. I.e., we add this edge (e, t) if there is some independent set I_i that remains independent if e is added to it.
- Add directed edge (e, f) for any distinct $e, f \in E$ such that $I_i + e \notin \mathcal{I}$ and $f \in C(I_i, e)$ for some i . That is, we add an edge (e, f) where e directs **to** the elements of a (nec. unique) circuit that is **potentially** created when e is added to I_i for some i .

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- If the shortest path is $S = (s, e, t)$ then we can add e to some independent set and it is still independent.

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- If the shortest path is $S = (s, e, t)$ then we can add e to some independent set and it is still independent.
- If the shortest path is $S = (s, e, f, t)$ then we can add e to some I_1 , create a circuit, but that gets broken when we remove f from that circuit rendering I_1 once again independent, but then there must be some other I_2 that f can be added to w/o making I_2 independent. Thus, the new independent sets are $I_1 + e - f$ and $I_2 + f$, thus we are making progress.

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- If the shortest path is $S = (s, e, f_1, f_2, t)$ then we can:
 - ① add e to some I_1 , thus making a circuit C_1 due to edge (e, f_1) .

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 - 1 add e to some I_1 , thus making a circuit C_1 due to edge (e, f_1) .
 - 2 subtract f_1 from I_1 , eliminating the circuit C_1 .

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 - ③ add f_1 to some I_2 , thus making a circuit C_2 due to edge (f_1, f_2) .

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 - ② subtract f_1 from I_1 , eliminating the circuit C_1 .
 - ③ add f_1 to some I_2 , thus making a circuit C_2 due to edge (f_1, f_2) .
 - ④ subtract f_2 from I_2 , eliminating the circuit C_2 .

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- If the shortest path is $S = (s, e, f_1, f_2, t)$ then we can:
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 - ② subtract f_1 from I_1 , eliminating the circuit C_1 .
 - ③ add f_1 to some I_2 , thus making a circuit C_2 due to edge (f_1, f_2) .
 - ④ subtract f_2 from I_2 , eliminating the circuit C_2 .
 - ⑤ add f_2 to some I_3 , not making a circuit due to edge (f_2, t) .

thus making progress.

Flow solution theorem

Thus, we have outlined the proof of one direction in the following theorem. When all matroids are the same $\forall i, M_i = M$ for some matroid, we have:

Theorem 4.2

There is an (s, t) path in the aforementioned graph iff the set of independent sets $(I_i : i \in J)$ can be grown by one element and still be a partition of some subset of E .

The other direction can be shown as a consequence of Theorem 4.1.

Exercise

Problem To Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathcal{R}_+^E$;
- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M , and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*
- By the above theorem, the existence of such an A will certify that $x(E)$ is maximal in $P_{\text{ind. set}}$, minimal in terms of $f(A) = r_M(A) - x(A)$, and thus most violated in terms of A .
- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of f .
- This will also run in polynomial time.

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- We keep a family of independent sets $(I_i : i \in J)$ and coefficients $(\lambda_i : i \in J)$ such that $\sum_{i \in J} \lambda_i = 1$ and $y = \sum_{i \in J} \lambda_i \mathbf{1}_{I_i}$.

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- We gradually build up y by adding new independent sets (and augmenting J), adding to the existing independent sets, and adjusting coefficients.

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- We keep a family of independent sets $(I_i : i \in J)$ and coefficients $(\lambda_i : i \in J)$ such that $\sum_{i \in J} \lambda_i = 1$ and $y = \sum_{i \in J} \lambda_i \mathbf{1}_{I_i}$.
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- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we'll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.

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- The algorithm starts with $y = 0$, $J = \{0\}$, $I_0 = \emptyset$, and $\lambda_0 = 1$.

Augmenting path theorem

Theorem 5.1

If there is a directed path from s to t in G , then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) \geq y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

We will prove this next time.

Augmenting path theorem consequences

Corollary 5.2

For any $x \in \mathbb{R}_+^E$, we have

$$\max (y(E) : y \leq x, y \in P_f) = \min (x(A) + f(E \setminus A) : A \subseteq E) \quad (16)$$

Note: this was not used in the theorem above, rather it is a consequence!

Proof.

- First, any $y \in P$ with $y \leq x$, and any $A \subset E$, we have

$$y(E) = y(A) + y(E \setminus A) \leq r(A) + x(E \setminus A) \quad (17)$$

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- Then there exists no such $y' \in P$ s.t. $y'(E) > y(E)$, and the digraph won't have a directed path from s to t (by the theorem).



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- Then there exists no such $y' \in P$ s.t. $y'(E) > y(E)$, and the digraph won't have a directed path from s to t (by the theorem).
- Then, there is a set A such that $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$, or that $y(E) = r(A) + x(E \setminus A)$, thus demonstrating equality.



Scratch Paper

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Sources for Today's Lecture

- Jack Edmonds, “Matroid Partition”, 1968.
- W. Cunningham, “Testing Membership in Matroid Polyhedra”, 1984
- E. Lawler, “Matroid Intersection Algorithms”, 1975.
- L. Schrijver, “Combinatorial Optimization”, 2003.
- Krogdahl, “A Combinatorial Base for some Optimal Matroid Intersection Algorithms”, 1974.