A Bag Of submodular Non-Examples

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This document is a (growing) collection of functions that might be suspected to be submodular, but are actually not. A set function $f : 2^V \to \mathbb{R}$ over subsets of a ground set V is submodular if for any $A, B \subseteq V$ it holds that

$$f(A \cup B) - f(A \cap B) \leq f(A) + f(B). \tag{1}$$

An alternative definition, also named "diminishing marginal returns", is that for any $A \subseteq B \subseteq V$, and $a \in V \setminus B$, a submodular function satisfies

$$f(A \cup a) - f(A) \geq f(B \cup a) - f(B). \tag{2}$$

Here and in the sequel, we simplify $A \cup \{a\}$ to $A \cup a$.

1 Support of random variables

Assume we have n random variables, $x_1, \ldots, x_n$, with indices in $V = \{1, \ldots, n\}$, that take values in a discrete set $\mathcal{L}$. Let p be their joint distribution. We define a function $f : 2^V \to \mathbb{N}$ on sets of variables that counts the number of joint states of the variables with support in the marginal distribution. Let $\mathcal{L}^k$ be the product space of $\mathcal{L}$, such that each $L \in \mathcal{L}^k$ is an ordered k-tuple $(\ell_1, \ldots, \ell_k)$ with entries $\ell_i \in \mathcal{L}$. Let furthermore, for a set A of indices, $\mathbf{x_A} = \{x_i | i \in A\}$ be the ordered set (vector) of variables with indices in $A \subseteq V$. For any set of indices, we define a count of states whose marginal probability is not zero:

$$f(A) = \left| \left\{ L \in \mathcal{L}^{|A|} \;\middle|\; \sum_{L' \in \mathcal{L}^{n-|A|}} p(\mathbf{x_A} = L, \mathbf{x}_{V \setminus A} = L') \; > 0 \right\} \right| \tag{3}$$

We also define $f(\emptyset) = 0$. Thanks to Jeff Bilmes for this function.

The properties of f depend on p. Let, for instance, p have support everywhere, i.e., $p(x_A = L) > 0$ for any $L \in \mathcal{L}^{|A|}$. For any sets $A \subset B$ with $a \in V \setminus B$, we then have $f(B) \geq f(A)$, and

$$\begin{align} f(A \cup a) - f(A) &= f(A)f(a) - f(A) \tag{4}\\ &= f(a)(f(A) - 1) \tag{5}\\ &\leq f(a)(f(B) - 1) \tag{6}\\ &= f(B \cup a) - f(B). \tag{7} \end{align}$$

a	1	2	3	4
b	1	1	1	[1,10]
c	[1,10]	[1,10]	[1,10]	1

Table 1: States where $p_1(\mathbf{x}) > 0$.

In this case, f is supermodular.

However, if there are strict dependencies so that p does not have support everywhere, then the properties of f change. As an example, define three random variables: x_a takes values in $\{1, \dots, 4\}$, x_b in $\{1, \dots, 10\}$, and x_c in $\{1, \dots, 10\}$. We then set

$$p_1(\mathbf{x}) = \begin{cases} 1/40 & \text{if } x_a < 4 \text{ and } x_b = 1 \\ 1/40 & \text{if } x_a = 4 \text{ and } x_c = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Table 1 lists the states with support. Let us check if f is sub- or supermodular if it uses p_1. For example,

$$\begin{aligned} f(a \cup \emptyset) - f(\emptyset) &= 4 \\ &< 21 = 31 - 10 = f(a \cup c) - f(c), \end{aligned}$$

violating submodularity. On the other hand,

$$\begin{aligned} f(a \cup b) - f(b) &= 13 - 10 \\ &> 0 = 40 - 40 = f(a \cup b \cup c) - f(b \cup c), \end{aligned}$$

violating supermodularity.

2 Logarithm of the support size

What about $\log f$, if f is defined as in Equation (3)? Thanks to Jeff Bilmes for this function, too.

If all variables have full joint support, then f is simply the product of the size of the state spaces, and $\log f$ a sum, i.e., a modular function.

Let us consider two other examples though, both over three variables (that means sets A and C have cardinality one), $x_a \in \{1, 2, 3, 4\}$, $x_b \in \{1, \dots, 10\}$, $x_c \in \{1, \dots, 10\}$. First, let

$$p_2(\mathbf{x}) = \begin{cases} 1/301 & \text{if } x_a < 4 \\ 1/301 & \text{if } x_a = 4 \text{ and } x_b = 1 \text{ and } x_c = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Then

$$\begin{aligned} \log f(a \cup c) - \log f(a) &= \log 31 - \log 4 < 2.05 \\ \log f(a \cup b \cup c) - \log f(a \cup b) &= \log 301 - \log 31 > 2.27 > \log f(a \cup c) - \log f(a). \end{aligned}$$

x_a	1	2	3	4
x_b	[1,10]	[1,10]	[1,10]	1
x_c	[1,10]	[1,10]	[1,10]	1

Table 2: Joint states where p_2 defined in Equation (9) is greater than zero.

Thus, $\log f$ for p_2 is not submodular. However, it is neither supermodular, since

$$\log f(b \cup c) - \log f(c) = \log 100 - \log 10 > 2.30$$
$$\log f(a \cup b \cup c) - \log f(a \cup c) = \log 301 - \log 31 < 2.28 < \log f(b \cup c) - \log f(c).$$

For illustration, Table 2 shows the possible joint states that the three variables can take.

3 Density of subgraphs

Given a graph $G = (V, E)$, let $e(S)$ be the number of edges interconnecting the nodes in $S \subseteq V$, i.e., $e(S) = |\{e = (u, v) \in E \mid u \in S, v \in S\}|$. Now define a function $f : 2^V \to \mathbb{R}$,

$$f(S) = \frac{e(S)}{|S|}. \tag{10}$$

Thanks to Reza Bosagh Zadeh for this function.

It is known that $e(S)$ is supermodular.

First, assume a graph G that has a complete bipartite subgraph (A, B, E') with $|A| = |B| = 10$, $e(A) = 0$, $e(B) = 0$, and $e(AunionB) = 100$. Let G have another node v that is connected to each node in A, but to no node in B. Then

$$f(A \cup v) - f(A) = \frac{10}{11} - 0 > 0.90$$
$$> 0.24 > \frac{110}{21} - \frac{100}{20} = f(A \cup B \cup v) - f(A \cup B).$$

This example satisfies the condition for submodularity. On the other hand, consider two disjoint single-node sets, $|A| = |B| = 1$, and a third node v that has an edge to B but not to A. Then

$$f(A \cup v) - f(A) = \frac{0}{2} - \frac{0}{1} = 0 < \frac{1}{3} - \frac{0}{2} = f(A \cup B \cup v) - f(A \cup B),$$

violating submodularity. Hence, in general this function f is neither submodular nor supermodular.